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Codes of split type

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Abstract

Generalizing a way to construct Golay codes, codes of *split type* are defined. A lot of interesting codes, for example, extremal codes of length $n \leq 40$ such as Golay codes and binary doubly even self-dual codes [48, 24, 12], [72, 36, w] with $w \geq 12$, are represented as codes of split type.

Key Words : linear code, Golay code, Mathieu group

1. Introduction

The MOG array was discovered by R.T. Curtis [4], and it is recognized to be one of the greatest object for investigations involving the Mathieu group M_{24}. In his paper [2] Ch.11, J.H.Conway gives a nice description for constructing the Golay codes. His works inspired us to generalizing Golay codes. As such a code, we introduce a code *of split type,* which is defined by the following way, i.e., let F'/F be an extension of finite fields, and let

$$I : F^n \longrightarrow F^{k \times n}, \ \ L : F^{k \times n} \longrightarrow (F')^n$$

be linear maps such that L has a set-theoretic section T. Here $F^{k \times n}$ is the space of $k \times n$ matrices with coefficient in F. For subsets $B \subset F^n$ and $D \subset (F')^n$, the linear code $C(B, D) = \langle I(B), \mathrm{T}(D) \rangle$ in $F^{k \times n}$ is called a *code of split type*. The binary and ternary Golay codes are in fact codes of split type. Fairly many codes are represented as codes of split type. We shall show some of them as examples.

The contents of this article is as follows. After we define codes of split type in §2, we shall discuss three kinds of codes of split type. A fundamental case is discussed in §3, and in the following §4 we shall unify the arguments developed in [6] Ch.5 and Ch.7, as a result, we shall obtain generalizations of Golay codes. Such a code has a criterion which characterizes codewords. In the case of the binary Golay code, such a criterion is called the Miracle Octad Generators. For each $n = 8l$ with $l \leq 5$, a binary *extremal* singly or doubly even and self-dual code with length n is represented as a code of split type.

In §5 generalizations $C_3(B, D)$ of Turyn's construction (cf. [2], Ch.11) will be given as a kind of codes of split type. Here we emphasize that our code $C_3(B, D)$ is a slight generalization of a cubic self-dual binary code given in [1]. In fact, a cubic code is defined for an $\mathbb{F}_4$ -linear code D, however our $C_3(B, D)$ is definable for an additive code D. We shall represent several binary or quaternary self-dual codes, for example, binary doubly-even self-dual codes [48, 24, 12] and [72, 36, w] with $w \geq 12$ as codes of split type.

A code $C(B, D)$ of split type has automorphisms induced by those of B and D. We shall discuss such automorphisms of codes of split type in §6. Lastly we shall touch M-matrices of codes of split type. We hope that a study of automorphisms and M-matrices of codes of split type contribute to further investigation, such as showing the uniqueness of the code [48, 24, 12], of codes of split type. T. Kondo ([12]) introduces M-matrices of the Steiner System $S(5, 8, 24)$ and uses them as a base for a story of Mathieu groups. In [9] and [11], we define M-matrices of the ternary Golay code and give applications of them. For further applications of M-matrices, we refer to [13], [8] and [10].

2. Codes of split type

Let p be a prime number. We denote by $\mathbb{F}_q$ the finite field with $q = p^f$ elements. The inner product of vectors

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$\mathbf{u} = (u_1, \cdots, u_n), \mathbf{v} = (v_1, \cdots, v_n) \in (\mathbb{F}_q)^n$ is defined by

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} u_i v_i^{\sigma},$$

where σ is the identity of $\mathbb{F}_q$ or the involution in $\mathrm{Aut}(\mathbb{F}_q)$ if it exists. For a vector $\mathbf{u} \in \mathbb{F}_q^n$, the weight $\mathrm{wt}(\mathbf{u})$ of $\mathbf{u}$ is the number of non-zero components of $\mathbf{u}$. A k-dimensional subspace of $(\mathbb{F}_q)^n$ is called an $[n, k]$ code over $\mathbb{F}_q$. For an $[n, k]$ code C, its dual $C^{\perp} = \{\mathbf{u} \in (\mathbb{F}_q)^n \mid (\mathbf{u}, \mathbf{v}) = 0 (\forall \mathbf{v} \in C)\}$ is an $[n, n-k]$ code. If $C \subset C^{\perp}$, C is said to be *self-orthogonal*, and if $C = C^{\perp}$, then C is said to be *self-dual*. For an $\mathbb{F}_p$-linear subspace C of $\mathbb{F}_q^n$, C is said to be *even* (resp. *singly-even*, *doubly-even*) if $\{\mathrm{wt}(\mathbf{u}) \mid \mathbf{u} \in C\} \subset 2\mathbb{Z}$ (resp. $\subset 2\mathbb{Z}$ and $\not\subset 4\mathbb{Z}$, $\subset 4\mathbb{Z}$). The minimum of the set $\{\mathrm{wt}(\mathbf{u}) \mid \mathbf{0} \neq \mathbf{u} \in C\}$ is called the *minimal distance* of C. An $[n, k]$ code with minimal distance d is called an $[n, k, d]$ code.

Now we shall define a code of split type. Let p be a prime number, and put $q = p^f$, $q' = p^{f'}$ with $f | f'$. For positive integers a and b, we denote by $(\mathbb{F}_q)^{a \times b}$ the space of $a \times b$ matrices with coefficients in $\mathbb{F}_q$. We fix a subset $\mathrm{K} = \{\omega_1, \cdots, \omega_k\}$ of the field $\mathbb{F}_{q'}$ satisfying

$$\text{(1)} \qquad \sum_{i=1}^{k} \omega_i = 0, \text{ and } \mathbb{F}_{q'} = \mathbb{F}_q(K).$$

Let

$$l : (\mathbb{F}_q)^{k \times 1} \longrightarrow \mathbb{F}_{q'}$$

be an F_q-linear map defined by

$$l({}^t(x_1, \cdots, x_k)) = \sum_{i=1}^{k} \omega_i x_i.$$

Then the direct sum

$$L = \oplus_{i=1}^{n} l : (\mathbb{F}_q)^{k \times n} \longrightarrow (\mathbb{F}_{q'})^n$$

is an $\mathbb{F}_q -$ linear map. Let

$$T : (\mathbb{F}_{q'})^n \longrightarrow (\mathbb{F}_q)^{k \times n}$$

be a set-theoretic section of L.

Let

$$\iota : \mathbb{F}_q \longrightarrow (\mathbb{F}_q)^{k \times n}, \quad \iota(x) = {}^t(x, \cdots, x),$$

and let

$$I = \oplus_{i=1}^{n} \iota : (\mathbb{F}_q)^n \longrightarrow (\mathbb{F}_q)^{k \times n}.$$

Then, by (1), we have $L \circ I = 0$.

If we define an $\mathbb{F}_q$-linear map $s : (\mathbb{F}_q)^{k \times 1} \longrightarrow \mathbb{F}_q$ by

$$s({}^t(x_1, \cdots, x_n)) = \sum_{i=1}^{k} x_i,$$

then $s \circ \iota(x) = kx$ for $x \in \mathbb{F}_q$. We denote by S the direct sum $\oplus^n s$:

$$S = \oplus^n s : (\mathbb{F}_q)^{k \times n} \longrightarrow (\mathbb{F}_q)^n.$$

For subsets $B \subset (\mathbb{F}_q)^n$ and $D \subset (\mathbb{F}_{q'})^n$, we define the linear code $C(B, D; I, T)$ by

$$C(B, D; I, T) = \langle I(B), T(D) \rangle_{\mathbb{F}_q} \subset (\mathbb{F}_q)^{k \times n}.$$

We call $C(B, D; I, T)$ the *code of split type* associated with B, D, I and T.

Collecting our notation together, we have the following:

$$\text{(2)} \qquad \begin{array}{ccccc} (\mathbb{F}_q)^n & \underset{S}{\overset{I}{\leftrightarrows}} & (\mathbb{F}_q)^{k \times n} & \underset{T}{\overset{L}{\leftrightarrows}} & (\mathbb{F}_{q'})^n, \\ \cup & & \cup & & \cup \\ B & & C(B, D; I, T) & & D \end{array}$$

$$S \circ I = k \cdot id_{(\mathbb{F}_q)^n}, \quad L \circ T = id_{(\mathbb{F}_{q'})^n}, \quad L \circ I = 0.$$

Here we give an interesting example of a code of split type.

Example 1 ([2] Ch.12) *Let P and L be the point code and the line code, respectively. They are isomorphic to the Hamming* [8, 4, 4] *code. Put*

$$K = \mathbb{F}_4^{\times} = \{\omega_1 = 1, \omega_2 = \omega, \omega_3 = \omega^2 = \bar{\omega}\}.$$

Define $s : \mathbb{F}_4 \longrightarrow (\mathbb{F}_2)^{3 \times 1}$ by

$$s(a\omega + b) = t((a + b), b, a), a, b \in \mathbb{F}_2.$$

Then $S = \oplus_{i=1}^{8} s$ is a section of the linear map $L : (\mathbb{F}_2)^{3 \times 8} \longrightarrow (\mathbb{F}_4)^8$. Put $B = P$, $D = L \otimes \mathbb{F}_4$.
Then $C(B, D; I, T)$ is the binary Golay code. This is so-called Turyn's construction.

3. Case I

According to choices of a set K or a section T, we obtain various codes of split type. In this section, we shall discuss in the following situation: Let the notation be as in §2, if the contrary is not stated. We take the field $\mathbb{F}_{q'}$ as a set $K = \{\omega_1 = 0, \omega_2, \cdots, \omega_k\}$; hence $k := |K| = q'$.

Define a section of the linear map L by the direct sum

$$T_1 = \oplus_{i=1}^{n} t : (\mathbb{F}_{q'})^n \longrightarrow (\mathbb{F}_q)^{k \times n},$$

of

$$t : \mathbb{F}_{q'} \longrightarrow (\mathbb{F}_q)^{k \times 1}, \quad \omega_i \longmapsto (\omega_i)$$

where (ω_i) is the column vector whose i-th component is 1 and the others are 0. As usual, let

$$\{\mathbf{e}_1 = (1, 0, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, \cdots, 0, 1)\}$$

be the standard basis of the vector space $(\mathbb{F}_q)^n$.

By the definitions, we have the following:

Lemma 1 *1.* $(I(\mathbf{e}_i), I(\mathbf{e}_j)) = |\mathbb{F}_{q'}|\delta_{ij} = 0$,

2. If $\mathbf{d} \in (\mathbb{F}_{q'})^n$, then $(I(\mathbf{e}_i), T_1(\mathbf{d})) = 1$,

3. $(T_1(\mathbf{d}), T_1(\mathbf{d}')) = n - \mathrm{wt}(\mathbf{d} - \mathbf{d}')$ *for vectors* $\mathbf{d}$ *and* $\mathbf{d}'$ of $(\mathbb{F}_{q'})^n$.

First we discuss in the following situation:

$$q = 2, q' = 4, K = \mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\},$$
$$B = \langle \mathbf{e}_1 - \mathbf{e}_2, \cdots, \mathbf{e}_1 - \mathbf{e}_n \rangle.$$

For an $\mathbb{F}_2$-linear subspace $D \subset \mathbb{F}_4^n$, let $C_1(D) = C(B, D; L, T_1)$ be a code of split type.

Lemma 2 *Assume that D is even. Then, for any vectors* $\mathbf{d}$ *and* $\mathbf{d}'$ *in D, we have*

$$T_1(\mathbf{d}) + T_1(\mathbf{d}') + T_1(0) + T_1(\mathbf{d} + \mathbf{d}') \in I(B).$$

If $\{\mathbf{d}_1, \cdots, \mathbf{d}_m\}$ *is a basis of D over* $\mathbb{F}_2$*, then the set*

$$X = \{I(\mathbf{e}_1 - \mathbf{e}_2), \cdots, I(\mathbf{e}_1 - \mathbf{e}_n), T_1(0), T_1(\mathbf{d}_1), \cdots, T_1(\mathbf{d}_m)\}$$

is a basis of $C_1(D)$ *over* $\mathbb{F}_2$*. In particular, we have*

$$\dim_{\mathbb{F}_2} C_1(D) = \dim_{\mathbb{F}_2}(D) + n.$$

Proof. We denote by I_n the set $\{1, 2, \cdots, n\}$. For

$$\mathbf{d} = (d_1, d_2, \cdots, d_n), \quad \mathbf{d}' = (d'_1, d'_2, \cdots, d'_n),$$

we define subsets a, b, c and c' of I_n by

$$a := \{i \in I_n \mid 0 \neq d_i \neq d'_i \neq 0\},$$
$$b := \{i \in I_n \mid d_i = d'_i \neq 0\},$$
$$c := \{i \in I_n \mid d_i \neq 0, d'_i = 0\},$$
$$c' := \{i \in I_n \mid d_i = 0, d'_i \neq 0\}.$$

Since D is even, it follows that $|a| + |b| + |c|$ and $|a| + |b| + |c'|$ are even; hence $|c| + |c'|$ is even. Here, $|a|$ is the cardinality of the set a, etc. On the other hand, $|\mathrm{supp}(\mathbf{d} + \mathbf{d}')| = |a| + |c| + |c'|$ is even. Thus $|a|$ is even. If $i \in a$, then $\{d_i, d'_i, d_i + d'_i\} = \{1, \omega, \bar{\omega}\}$ and the i-th column of

$$A := T_1(\mathbf{d}) + T_1(\mathbf{d}') + T_1(0) + T_1(\mathbf{d} + \mathbf{d}')$$

is ${}^t(1, 1, 1, 1)$. If $i \notin a$, then one of the following three cases holds:

$$1.\ d_i = d'_i, \quad 2.\ d'_i = 0, \quad 3.\ d_i = 0.$$

Therefore the i-th column of A is ${}^t(0, 0, 0, 0)$. Hence

$$A = \sum_{i \in a} I(\mathbf{e}_i) \in I(B).$$

Thus we obtain the first part of the Lemma. In particular, the set X generates the code $C_1(D)$.

Now we shall show that X is linearly independent over $\mathbb{F}_2$. Set $\mathbf{b}_i = \mathbf{e}_1 - \mathbf{e}_i$. Suppose that

$$\sum_{i=2}^{n} \beta_i I(\mathbf{b}_i) + \sum_{j=0}^{m} \delta_j T(\mathbf{d}_j) = 0 \quad (\mathbf{d}_0 = \mathbf{0}))$$

with $\beta_i, \delta_j \in \mathbb{F}_2$. Since $L \circ T_1 = id_{(\mathbb{F}_{q'})^n}$ and $L \circ I = 0$, applying the linear map L for this equation, we get

$$\sum_{j=1}^{m} \delta_j \mathbf{d}_j = 0.$$

Therefore $\delta_1 = \cdots = \delta_m = 0$. Thus we have

$$\sum_{i=2}^{n} \beta_i I(\mathbf{b}_i) + \delta_0 T(\mathbf{0}) = 0.$$

If $\delta_0 \neq 0$ then we have $T_1(\mathbf{0}) = I(\mathbf{b})$ for some $\mathbf{b} \in B$. By the definition of I, this is absurd; hence $\delta_0 = 0$. Therefore we have

$$\sum_{i=2}^{n} \beta_i I(\mathbf{b}_i) = 0.$$

Since I is injective, we get $\beta_2 = \cdots = \beta_n = 0$. Thus X is linearly independent over $\mathbb{F}_2$; hence we have the last assertion. □

Proposition 3 *Let* $q = 2$ and $q' = 4$ *and assume that* $n \geq 4$ *and* n *is even. Let* D *be an* n*-dimensional* $\mathbb{F}_2$*-subspace of* $\mathbb{F}_4^n$ *such that each element in* D *is even weight. If* $n \equiv 0$ (resp.2) (mod 4), *then the code* $C_1(D)$ *is a binary, self-dual and doubly (resp. singly) even* $[4n, 2n, d]$ *code with* $d \geq 4$. *Further if* $n \geq 8$ (*resp.* $n = 6$) *and the minimal distance of* D *is greater than or equal to* 4, *then* $C_1(D)$ *is a* $[4n, 2n, 8]$ (resp.[24, 12, 6])-*code*.

Proof. By Lemma 1 and Lemma 2, $C_1(D)$ is a $2n$-dimensional self-dual code. Any codeword of $C_1(D)$ is written in the following forms:

$$I(\mathbf{b}),\ \ T_1(\mathbf{d}),\ \ I(\mathbf{b}) + T_1(\mathbf{d}),\ \ I(\mathbf{b}) + T_1(\mathbf{0}) + T_1(\mathbf{d}) \qquad (\mathbf{b} \in B, \mathbf{d} \in D).$$

Then we have

$$\begin{aligned} &\mathrm{wt}(I(\mathbf{b})) = 4\mathrm{wt}(\mathbf{b}),\ \mathrm{wt}(T_1(\mathbf{d})) = n, \\ &\mathrm{wt}(I(\mathbf{b}) + T_1(\mathbf{d})) = n - \mathrm{wt}(\mathbf{b}) + 3\mathrm{wt}(\mathbf{b}) \\ &\qquad = n + 2\mathrm{wt}(\mathbf{b}). \end{aligned}$$

Moreover, $\mathrm{wt}(I(\mathbf{b}) + T_1(\mathbf{0}) + T_1(\mathbf{d}))$ is equal to

$$4|\mathrm{supp}(\mathbf{b}) - \mathrm{supp}(\mathbf{d})| + 2|\mathrm{supp}(\mathbf{b}) \cap \mathrm{supp}(\mathbf{d})| + 2|\mathrm{supp}(\mathbf{d}) - \mathrm{supp}(\mathbf{b})|.$$

Since D is even, it follows that this is a multiple of 4. By these calculations, we get our assertions. □

Example 2 (Binary singly-even self-dual [24, 12, 6] code) *Let* $D = \mathcal{H}$ *be the Hexacode,i.e.*,

$$\mathcal{H} = \{(a, b, c, \phi(1), \phi(\omega), \phi(\bar{\omega})\,|\, a, b, c \in \mathbb{F}_4,\ \phi(x) = ax^2 + bx + c\} \subset (\mathbb{F}_4)^6.$$

Then $C_1(\mathcal{H})$ *is a binary, singly even and self-dual* [24, 12, 6] *code with weight distribution:*

weight	0	6	8	10	12	14	16	18	24
#	1	64	375	960	1296	960	375	64	1

Example 3 (Binary doubly-even self-dual [32, 16, 8] code) *Let* H_8 *be the Hamming code and* $D = H_8 \otimes \mathbb{F}_4$. *Then* $C_1(D)$ *is a binary, doubly even and self-dual* [32, 16, 8] *code with weight distribution:*

weight	0	8	12	16	20	24	32
#	1	620	13888	36518	13888	620	1

Example 4 (Binary singly-even self-dual [40, 20, 8] code) *Let* D *be an even* [10, 5, 4] *code over* $\mathbb{F}_4$. *For example, if* D *is generated by*

$$\begin{aligned} &d_1 = 1111000000,\ \ d_2 = 0011110000, \\ &d_3 = 0000111100,\ \ d_4 = 0000001111, \\ &d_5 = 1212121212 \end{aligned}$$

where $2 = \omega$. *Then,* $C_1(D)$ *is a binary singly even self-dual* [40, 20, 8] *code with weight distribution:*

weight	0	8	10	12	14	16	18
#	1	285	1024	11040	46080	117090	215040

20	22	24	26	28	30	32	40
267456	215040	117090	46080	11040	1024	285	1

Secondly, we are in the following situation:

$$q = q' = 3,\ \ B = \langle \mathbf{e}_1 - \mathbf{e}_2, \cdots, \mathbf{e}_1 - \mathbf{e}_n \rangle.$$

and I, L, T_1 are the same as before.

By the same way as the above, we have the following, so we shall omit their proofs:

Lemma 4 *Assume that each codeword* $\mathbf{d}$ *of* D *satisfies* $\mathrm{wt}(\mathbf{d}) \equiv 0 \pmod 3$. *Then, for any* $\mathbf{d}$ *and* $\mathbf{d}'$ *from D, we have*

$$T_1(\mathbf{d}) + T_1(-\mathbf{d}) + T_1(0),\ T_1(\mathbf{d}) + T_1(\mathbf{d}') + T_1(0) + T_1(\mathbf{d} + \mathbf{d}') \in I(B).$$

In particular, if $\{\mathbf{d}_1, \cdots, \mathbf{d}_m\}$ *is a basis of D, then the set* $X = \{I(\mathbf{e}_1 - \mathbf{e}_2), \cdots, I(\mathbf{e}_1 - \mathbf{e}_n), T_1(\mathbf{0}), T_1(\mathbf{d}_1), \cdots, T_1(\mathbf{d}_m)\}$ *is a basis of the code* $C_1(D)$ *over* $\mathbb{F}_3$*;*

$$\dim_{\mathbb{F}_3} C_1(D) = \dim_{\mathbb{F}_3}(D) + n.$$

Proposition 5 *Let* $q = q' = 3$ *and assume that* $n \equiv 0$ (mod 6). *Let* D *be an* $n/2$*-dimensional code in* $(\mathbb{F}_2)^n$ *satisfying* $\mathrm{wt}(\mathbf{d}) \equiv 0 \pmod 3$ *for any* $\mathbf{d} \in D$. *Then* $C_1(D)$ *is a* $[3n, 3n/2, 6]$*-code*.

4. Case II

In this section we shall unify the arguments developed in [6] Ch.5 and Ch.7 and give a generalization of Golay codes. Our situation is the same as (2), and assume

(A) $q = 2, q' = k = 4; n$: even

or

(B) $q = q' = k = 3$; $n \equiv -2 \pmod 6$.

Set $\mathbb{F}_4 = \{\omega_1 = 0, \omega_2 = 1, \omega_3 = \omega, \omega_4 = \omega^2\}$ and $\mathbb{F}_3 = \{\omega_1 = 0, \omega_2 = 1, \omega_3 = -1\}$.

By the definition, $L(I(\mathbf{e}_1)) = 0$; hence the map

$$T_2 : (\mathbb{F}_{q'})^n \longrightarrow (\mathbb{F}_q)^{k \times n}$$

defined by $T_2(\mathbf{d}) = T_1(\mathbf{d}) + I(\mathbf{e}_1)$ is also a section of the linear map $L : (\mathbb{F}_q)^{q' \times n} \longrightarrow (\mathbb{F}_{q'})^n$.

For a linear code B in $(\mathbb{F}_q)^n$ and an $\mathbb{F}_q$-linear subspace D of $(\mathbb{F}_{q'})^n$, we denote by $C_2(B, D)$ the code $C(B, D; I, T_2)$ of split type. The following lemmas are proved by the same way as in the previous section:

Lemma 6

$$(T_2(\mathbf{d}), T_2(\mathbf{d}')) = 2 + n - \mathrm{wt}(\mathbf{d} - \mathbf{d}') \qquad (\mathbf{d}, \mathbf{d}' \in (\mathbb{F}_{q'})^n).$$

Lemma 7 *If $\{\mathbf{b}_1, \cdots, \mathbf{b}_l\}$ is a linearly independent subset of B and $\{\mathbf{d}_1, \cdots, \mathbf{d}_m\}$ is a linearly independent subset of D over $\mathbb{F}_q$, then the set*

$$\{I(\mathbf{b}_1), \cdots, I(\mathbf{b}_l), T_2(\mathbf{d}_1), \cdots, T_2(\mathbf{d}_m), T_2(\mathbf{0})\}$$

is linearly independent over $\mathbb{F}_q$. In particular,

$$\dim_{\mathbb{F}_q} C_2(B, D) \geq \dim_{\mathbb{F}_q}(B) + \dim_{\mathbb{F}_q}(D) + 1.$$

Theorem 8 *Assume that the following conditions are satisfied:*$\oplus_{i=1}^{n} l$

(C1) $\sum_{i=1}^{n} b_i = 0$ *for every* $\mathbf{b} = (b_1, \cdots, b_n) \in B$,

(C2) $\mathrm{wt}(\mathbf{d}) \equiv n + 2 \pmod p$ *for every* $\mathbf{d} \in$ D.

Then we have

$$\dim_{\mathbb{F}_q}(B) + \dim_{\mathbb{F}_q}(D) + 1 \leq nk/2. \tag{3}$$

If the equality (3) *holds, for example* $\dim_{\mathbb{F}_q}(B) = n - 1$ and

$$\dim_{\mathbb{F}_q}(D) = \frac{n(k-2)}{2}, \tag{4}$$

then the code $C_2(B, D)$ is self-dual and

$$\dim_{\mathbb{F}_q}(C_2(B, D)) = nk/2 = \dim_{\mathbb{F}_q}(B) + \dim_{\mathbb{F}_q}(D) + 1.$$

In this case, if $\{\mathbf{b}_i | 1 \leq i \leq \dim_{\mathbb{F}_q}(B)\}$ is a basis of B and $\{\mathbf{d}_j | 1 \leq j \leq \dim_{\mathbb{F}_q}(D)\}$ is a basis of D, then the set $\{I(\mathbf{b}_i)$, $T_2(\mathbf{d}_j)$, $T_2(\mathbf{0}) | 1 \leq i \leq \dim_{\mathbb{F}_q}(B)$, $1 \leq j \leq \dim_{\mathbb{F}_q}(D)\}$ forms a basis of $C_2(B, D)$.

Proof. We denote $C_2(B, D)$ by C. By Lemma 6 and (C1), (C2), we have

1. $(I(\mathbf{b}), I(\mathbf{b}')) = 0 \ (\forall \mathbf{b}, \mathbf{b}' \in (\mathbb{F}_q)^n)$,
2. $(I(\mathbf{b}), T_2(\mathbf{d})) = \sum_{i=1}^{n} b_i = 0 \ (\forall \mathbf{b} \in B, \mathbf{d} \in (\mathbb{F}_{q'})^n)$,
3. $(T_2(\mathbf{d}), T_2(\mathbf{d}')) = n + 2 - \mathrm{wt}(\mathbf{d} - \mathbf{d}') = 0 \ (\forall \mathbf{d}, \mathbf{d}' \in (\mathbb{F}_{q'})^n)$.

Hence we have $C \subset C^{\perp}$. By Lemma 7, we have

$$\begin{aligned} &\dim_{\mathbb{F}_q}(B) + \dim_{\mathbb{F}_q}(D) + 1 \\ &\quad \leq \dim_{\mathbb{F}_q}(C) \\ &\quad \leq \dim_{\mathbb{F}_q}(C^{\perp}) \\ &\quad \leq nk - (\dim_{\mathbb{F}_q}(B) + \dim_{\mathbb{F}_q}(D) + 1). \end{aligned} \tag{5}$$

Thus we have

$$\dim_{\mathbb{F}_q}(B) + \dim_{\mathbb{F}_q}(D) + 1 \leq nk/2.$$

If the equality holds here, by (5), we have

$$C = C^{\perp}, \dim_{\mathbb{F}_q}(C) = kn/2.$$

□

Now we fix a code

$$B = \langle \mathbf{e}_1 - \mathbf{e}_i \,|\, 2 \leq i \leq n \rangle.$$

Then B satisfies the condition (C1). In the case (A), let D be an even $\mathbb{F}_2$-linear subspace of $(\mathbb{F}_4)^n$ of dimension n. On the other hand, in the case (B), let D be an $\mathbb{F}_3$-linear code in $(\mathbb{F}_3)^n$, with $\dim_{\mathbb{F}_3}(D) = n/2$, such that $\{\mathrm{wt}(\mathbf{d}) \,|\, \mathbf{d} \in D\} \subset 3\mathbb{Z}$. Then the code D satisfies the condition (C2) and (4). We denote by $C_2(D)$ the code $C(B, D; I, T_2)$ of split type. By Theorem 8, $C_2(D)$ is self-dual and

$$\dim_{\mathbb{F}_q}(C_2(D)) = nk/2 = \dim_{\mathbb{F}_q}(B) + \dim_{\mathbb{F}_q}(D) + 1.$$

The codewords of the code $C_2(D)$ are determined in the following way: this is no less a criterion than MOG or MINIMOG (cf. [2] Ch. 11).

Theorem 9 *Under the above situation, the following hold:*

$$\begin{aligned} C_2(D) &= \langle I(B), T_1(0) \rangle^{\perp} \cap L^{-1}(D) \\ &= \{x = (x_{ij}) \in (\mathbb{F}_q)^{k \times n} \,| \\ &\qquad \sum_{i=1}^{k} x_{il} = -\sum_{j=1}^{n} x_{1j} (\forall l), L(x) \in D\}. \end{aligned}$$

Proof. Let

$$W = \langle I(B), T_1(\mathbf{0})\rangle^{\perp}.$$

By lemma 7, we have $\dim_{\mathbb{F}_q}(W) = nq' - (\dim_{\mathbb{F}_q}(B) + 1) = n(k-1)$. Consider the restriction of the linear map L to W:

$$L|_W : W \longrightarrow \mathbb{F}_{q'}^n$$

We shall show that $L|_W$ is surjective. Take any $\mathbf{d} \in \mathbb{F}_{q'}^n$, then, by lemma 1, $(T_2(\mathbf{d}), T_2(\mathbf{0})) = n + 2 - \mathrm{wt}(\mathbf{d})$. Therefore if we set $w = T_2(\mathbf{d}) + aI(\mathbf{e}_1)$ with $a = \mathrm{wt}(\mathbf{d}) - n - 2$, then we have, by Lemma 6,

$$\begin{aligned}(w, T_2(\mathbf{0})) &= (T_2(\mathbf{d}) + aI(\mathbf{e}_1), T_2(\mathbf{0}))\\ &= n + 2 - \mathrm{wt}(\mathbf{d}) + a = 0.\end{aligned}$$

Let $\mathbf{b} = (b_1, \cdots, b_n) \in B$, then, by the condition (C1), we have $(w, I(\mathbf{b})) = (T_2(\mathbf{d}), I(\mathbf{b})) = \sum_{i=1}^n b_i = 0$. Thus w is contained in W and $L(w) = L(T_2(\mathbf{d}) + aI(\mathbf{e}_1)) = \mathbf{d}$; hence $L|_W$ is surjective.

We denote by U the kernel of the linear map $L|_W$. Since

$$\begin{aligned}&\dim_{\mathbb{F}_q}(W) = nk - (\dim_{\mathbb{F}_q}(B) + 1),\\ &\dim_{\mathbb{F}_q}(\mathbb{F}_{q'}^n) = |\mathbb{F}_{q'} : \mathbb{F}_q|n,\end{aligned}$$

it follows

$$\begin{aligned}\dim_{\mathbb{F}_q}(U) &= nk - (\dim_{\mathbb{F}_q}(B) + 1) - |\mathbb{F}_{q'} : \mathbb{F}_q|n\\ &= n(k - 1 - f') = n.\end{aligned}$$

Since $C_2(D)$ is self-dual, it follows $C_2(D) \subset (L|_W)^{-1}(D)$. Thus, by Theorem 8 we have

$$\begin{aligned}\dim_{\mathbb{F}_q}((L|_W)^{-1}(D)) &= \dim_{\mathbb{F}_q}(U) + \dim_{\mathbb{F}_q}(D)\\ &= n + \frac{n(k-2)}{2} = nk/2.\end{aligned}$$

Therefore we have $C_2(D) = (L|_W)^{-1}(D)$. Thus we have shown the first equality.

For a matrix $x = (x_{ij}) \in (\mathbb{F}_q)^{k \times n}$, we have

$$(x, I(\mathbf{e}_1) - I(\mathbf{e}_l)) = \sum_{i=1}^k x_{i1} - \sum_{i=1}^k x_{il}$$

and

$$(x, T_2(\mathbf{0})) = (x, I(\mathbf{e}_1) + T(\mathbf{0})) = \sum_{i=1}^k x_{i1} + \sum_{j=1}^n x_{1j}.$$

Therefore, by the first equality, we have the second one. □

Example 5 (The binary Golay code) *Let* $q = 2$, $q' = 4$, $n = 6$, *and* D *the Hexacode* $\mathcal{H}$. *Then* $C_2(\mathcal{H})$ *is the extended binary Golay code* [24, 12, 8] *with weight distribution:*

weight	0	8	12	16	24
#	1	759	2576	759	1

Example 6 (The ternary Golay code) *Let* $q = q' = 3$, $n = 4$ *and* D *the Tetracode* $\mathcal{T}$, *i.e,*

$$\mathcal{T} = \{(a, b, \phi(1), \phi(-1)) \mid a, b \in \mathbb{F}_3, \phi(x) = ax + b\} \subset \mathbb{F}_3^4$$

Then $C_2(\mathcal{T})$ *is the extended ternary Golay code.*

Example 7 (Hamming code) *Let* $q = 2$, $q' = 4$, $n = 2$ *and let* D *be a code in* $\mathbb{F}_4^2$ *spanned by* $\langle(1, 1)\rangle$ or $\langle(1, \omega)\rangle$ *over* $\mathbb{F}_4$. *Then* $C_2(D)$ *is the Hamming* [8, 4, 4] *code.*

Now we shall investigate the minimal distance of a code $C_2(D)$.

Theorem 10 *Assume that* $q = 2$ *and* $q' = 4$. *Suppose* $n \geq 6$ *and* n *is even. Furthermore, assume* D *is an even* n *dimensional* $\mathbb{F}_2$*-linear subspace with minimal distance* $w \geq 4$. *Then the code* $C_2(D)$ *is a self-dual binary* $[4n, 2n, 8]$ *code. Moreover, if* $n \equiv 2$ *(resp.* 0*)* (mod 4), *then* $C_2(D)$ *is doubly (resp. singly) even.*

Proof. Let d be the minimal distance of $C_2(D)$. Since the weight of $I(\mathbf{e}_1) + I(\mathbf{e}_2) \in C_2(D)$ is 8, it follows $d \leq 8$. Let $x = (x_{ij})$ be an element in $C_2(D)$ with $\mathrm{wt}(x) = d$. By Theorem 9, the value $\sum_{i=1}^k x_{ij} = -\sum_{j=1}^n x_{1j}$ does not depend on the choice of j. We call this value the parity of x and denote by $\mathrm{parity}(x)$. If $\mathrm{parity}(x) = 1$, then x has at least one point in each column. Hence if $n \geq 8$, $d = \mathrm{wt}(x) \geq 8$. In the case when $n = 6$, if x has one point in each column, by $\mathrm{parity}(x) = 1$, then the weight of $L(x) \in D$ has to be odd. Since D is even, this cannot occur. Therefore $d = \mathrm{wt}(x) \geq 8$. Assume x has parity 0. Then if the coordinate of $L(x)$ at i is not 0, then the number of non-zero components of x in the i−th column is two. Therefore $\mathrm{wt}(x) = 2\mathrm{wt}(L(x)) + 4a \geq 8$. Here a is the number of columns in which every components of x is 1.

Thus we have proved the former.

Now we shall prove the last assertion. If $x \in C_2(D)$ has even parity, then we already know that $\mathrm{wt}(x) \equiv 0 \pmod 4$. Assume that x has odd parity. Set

1. b_1 is the number of columns i where $x_{1i} = 1$ and two of $\{x_{2i}, x_{3i}, x_{4i}\}$ are 1,
2. b_2 is the number of columns i where one of $\{x_{2i}, x_{3i}, x_{4i}\}$ is 1,
3. b_3 is the number of columns i where $x_{1i} = 1$ and all of $\{x_{2i}, x_{3i}, x_{4i}\}$ are 0,
4. b_4 is the number of columns i where $x_{1i} = 0$ and two of $\{x_{2i}, x_{3i}, x_{4i}\}$ are 1.

Since $L(x)$ has even weight, $b_1 + b_2$ has to be even. By $\mathrm{parity}(x) = 1$, $b_1 + b_3$ is odd. Therefore $b_2 + b_3$ is odd; hence $b_1 + b_4 = n - (b_2 + b_3)$ is odd. Thus we have

$$\mathrm{wt}(x) = n + 2(b_1 + b_4) \equiv n + 2 \pmod 4.$$

Hence we have the last assertion. □

Example 8 (Binary singly-even self-dual [32, 16, 8] code) *Let $q = 2$, $q' = 4$. Let $D \subset \mathbb{F}_4^8$ be a singly even self-dual* [8, 4, 4] *code over $\mathbb{F}_4$ dened by*

$$\mathrm{D} = \{(a, b, c, d, b + c + d, a + c + d, a + b + d, a + b + c) \mid a, b, c, d \in \mathbb{F}_4\}.$$

Then $C_2(D)$ is a binary singly even self-dual [32, 16, 8] *code with weight distribution:*

weight	0	8	10	12	14	16
#	1	364	2048	6720	14336	18598

18	20	22	24	32
14336	6720	2048	364	1

Example 9 (Binary doubly-even self-dual [40, 20, 8] code) *Let D be the even* [10, 5, 4] code over $\mathbb{F}_4$ *in* Example 4. *Then $C_2(D)$ is a binary doubly even self-dual* [40, 20, 8] *code with weight distribution:*

weight	0	8	12	16	20
#	1	285	21280	239970	525504

24	28	32	40
239970	21280	285	1

Similarly, we have the following:

Theorem 11 *Let $q = q' = 3$, and $n \geq 4$ with $n \equiv -2 \pmod 3$. Assume that D is a ternary $[n, n/2, w]$ code which satisfies $\mathrm{wt}(d) \equiv 0 \pmod 3$ for each $d \in D$ with $w \geq 3$. Then $C_2(D)$ is a ternary self-dual* $[3n, 3n/2, 6]$-*code.*

5. Case III

We are still in the situation (2) of §2. Furthermore, let $\mathbb{F}_{q_0}$ be a subfield of $\mathbb{F}_q$ and assume that the section T of L is an $\mathbb{F}_{q_0}$-linear map.

For an $\mathbb{F}_q$-subspace $B \subset (\mathbb{F}_q)^n$ and an $\mathbb{F}_{q_0}$-subspace $D \subset (\mathbb{F}_{q_0})^n$, let $C(B, D; I, T)$ be a code of split type.

Lemma 12 *Assume that $T(D) \subset (\mathbb{F}_{q_0})^{k \times n}$. If $\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_m$ are linearly independent vectors in D over $\mathbb{F}_{q_0}$, then $T(\mathbf{d}_1), T(\mathbf{d}_2), \cdots, T(\mathbf{d}_m)$ are linearly independent over $\mathbb{F}_q$.*

Proof. Since T is an injective $\mathbb{F}_{q_0}$-linear map, it follows that $T(\mathbf{d}_1), \cdots, T(\mathbf{d}_m)$ are linealy independent, in $(\mathbb{F}_{q_0})^{k\times n}$, over $\mathbb{F}_{q_0}$. This means that the rank of the matrix consisting of the vectors $T(\mathbf{d}_1), \cdots, T(\mathbf{d}_m)$ in $(\mathbb{F}_{q_0})^{kn} (\simeq (\mathbb{F}_{q_0})^{k\times n})$ is m. Therefore they are linearly independent over $\mathbb{F}_q$. □

Lemma 13 *Assume that $T(D) \subset (\mathbb{F}_{q_0})^{k\times n}$, $S \circ T = 0$ and $k \in \mathbb{F}_p^\times$. Then*

$$\dim_{\mathbb{F}_q} C(B, D; I, T) = \dim_{\mathbb{F}_q} B + \dim_{\mathbb{F}_{q_0}} D.$$

Proof. Let $\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_l\}$ be a basis for the code B over $\mathbb{F}_q$ and $\{\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_m\}$ be a basis for the subspace D over $\mathbb{F}_{q_0}$. Then the set $\{I(\mathbf{b}_i);\ T(\mathbf{d}_l)\}$ is linearly independent over $\mathbb{F}_q$. In fact, if they satisfy

$$\sum_{i=1}^{l} \alpha_i I(\mathbf{b}_i) + \sum_{j=1}^{m} \beta_j T(\mathbf{d}_j) = 0 \quad (\alpha_i, \beta_j \in \mathbb{F}_q),$$

then, applying the $\mathbb{F}_q$-linear map S, we obtain

$$k \sum_{i=1}^{l} \alpha_i \mathbf{b}_i = 0.$$

Here we used the relation $S \circ I = k \cdot id_{(\mathbb{F}_q)^n}$ and $S \circ T = 0$. Since the set $\{\mathbf{b}_i\}$ is a basis and $k \in \mathbb{F}_p^\times$, it follows that $\alpha_1 = \alpha_2 = \cdots = \alpha_l = 0$. Therefore we have

$$\sum_{j=1}^{m} \beta_j T(\mathbf{d}_j) = 0.$$

Now applying Lemma 12, we have $\beta_1 = \beta_2 = \cdots = \beta_m = 0$. Therefore the set $\{I(\mathbf{b}_i),\ S(\mathbf{d}_j)\}$ is linearly independent.

Any element $\mathbf{d} \in D$ can be written in the form

$$\mathbf{d} = \sum_{j=1}^{m} \gamma_j \mathbf{d}_j \quad (\gamma_j \in \mathbb{F}_{q_0}).$$

Since T is $\mathbb{F}_{q_0}$-linear, we have

$$T(\mathbf{d}) = \sum_{j=1}^{m} \gamma_j T(\mathbf{d}_j).$$

Therefore the set $\{T(\mathbf{d}_1), \cdots, T(\mathbf{d}_m)\}$ generates the subspace $\langle T(D)\rangle$; hence the set

$$\{I(\mathbf{b}_1), \cdots, I(\mathbf{b}_l), T(\mathbf{d}_1), \cdots, T(\mathbf{d}_m)\}$$

forms a basis of $C(B, D; I, T)$ over $\mathbb{F}_q$.

From now on we assume the following:

$$q_0 = q = 2^f,\ q' = 2^{2f} \text{ or } q_0 = 2^f,\ q = q' = 2^{2f}$$

and

$$\begin{aligned}
&\mathbb{F}_{q'} = \mathbb{F}_{q_0}(\epsilon),\\
&\epsilon^2 + \alpha\epsilon + \beta = 0 \quad (\alpha, \beta \in \mathbb{F}_{q_0}),\\
&K = \{\omega_1 = \beta, \omega_2 = \alpha\epsilon, \omega_3 = \epsilon^2\}.
\end{aligned}$$

Then neither α nor β are 0 and any element of $\mathbb{F}_{q'}$ can be written as $a\alpha\epsilon + b\beta$ with $a, b \in \mathbb{F}_{q_0}$. Define a section $T_3 : (\mathbb{F}_{q'})^n \longrightarrow (\mathbb{F}_q)^{3\times n}$, of the linear map $L : (\mathbb{F}_q)^{3\times n} \longrightarrow (\mathbb{F}_{q'})^n$, by the sum $\oplus^n t$, where

$$t : \mathbb{F}_{q'} \longrightarrow (\mathbb{F}_q)^{3\times 1},\ a\alpha\epsilon + b\beta \longmapsto \begin{pmatrix} a \\ b \\ a+b \end{pmatrix}.$$

Then T_3 is an $\mathbb{F}_{q_0}$-linear map and the composition of the summation map $S : \mathbb{F}_q^{3\times n} \longrightarrow \mathbb{F}_q$ and T_3 is 0:

$$S \circ T_3 = 0. \tag{6}$$

Recall that the linear map L is the sum $\oplus^n l$ where

$$l : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto x_1\beta + x_2\alpha\epsilon + x_3\epsilon^2.$$

If $q' = 4$ and $\mathbb{F}_2 = \{0, 1\}$, then $\mathbb{F}_4 = \mathbb{F}_2(\omega)$ where ω is a root of $x^2 + x + 1$, and the linear map $t : \mathbb{F}_4 \longrightarrow \mathbb{F}_2^{3\times 1}$ is given by

$$1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ \omega \longmapsto \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ \bar{\omega} \longmapsto \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{7}$$

If $q' = 16$ and $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, then $\mathbb{F}_{16} = \mathbb{F}_4(\epsilon)$ where ϵ is a root of $x^2 + \omega x + 1 = 0$; hence $\alpha = \omega$, $\beta = 1$.

We denote by $C_3(B, D)$ the code $C(B, D; I, T_3)$ of split type.

Theorem 14 *If B is an $[n, k]$ code over $\mathbb{F}_q$ and D is an $[n, k']$ code over $\mathbb{F}_{q'}$, then $C_3(B, D)$ is a $[3n, k + 2k']$ code over $\mathbb{F}_q$.*

Proof. By the equality $\dim_{\mathbb{F}_{q_0}}(D) = 2\dim_{\mathbb{F}_{q'}}(D)$ and Lemma 13 we obtain the theorem. □

The above theorem contains the Turyn construction as a special case.

Example 10 *Set $q_0 = q = p^f$, $q' = p^{2f}$ and take α, β and ϵ as above. Let B and D be $[n, k]$ and $[n, k']$ codes over $\mathbb{F}_q$, then the code $C_3(B, D \otimes \mathbb{F}_{q'})$ is a code obtained by the Turyin construction.*

Example 11 *The following codes are discussed in Conway, Lomonaco and Sloane* [3]. *Let B be the $[5, 2, 4]$ code over $\mathbb{F}_4$ obtained by shortening the $[6, 3, 4]$ hexacode H over $\mathbb{F}_4$. A generator matrix for B is as follows:*

$$\begin{pmatrix} \omega & \bar{\omega} & \bar{\omega} & \omega & 0 \\ 0 & \omega & \bar{\omega} & \bar{\omega} & \omega \end{pmatrix}.$$

Let D be the conjugate of B, then $D_1 := C_3(B, D \otimes \mathbb{F}_{16})$ is a $[15, 6, 8]$ code over $\mathbb{F}_4$. Moreover, let B_1 be the $[15, 1, 15]$ binary code generated by $(1, \cdots, 1)$, then $C_3(B_1, D_1)$ is a $[45, 13, 16]$ binary code.

Now we shall study the self-duality of the code $C_3(B, D)$ with $q' = 4$, so we assume $q' = 4$ from now on.

Lemma 15 *1.* $(I(\mathbf{b}), I(\mathbf{b}')) = (\mathbf{b}, \mathbf{b}')$ $\quad (\mathbf{b}, \mathbf{b}' \in (\mathbb{F}_q)^n)$.

2. $(I(\mathbf{b}), \mathrm{T}_3(\mathbf{d})) = 0$, $(\mathbf{b} \in (\mathbb{F}_q)^n, \mathbf{d} \in (\mathbb{F}_4)^n)$.

3. In the space $(\mathbb{F}_2)^{3\times 1}$,

$$(t(k), t(k)) = 0,\ (t(k), t(k')) = 1 \qquad (k, k'(\neq) \in K = \mathbb{F}_4^{\times}).$$

4. *If* $\mathrm{wt}(\mathbf{d})$, $\mathrm{wt}(\mathbf{d}')$ and $\mathrm{wt}(\mathbf{d}-\mathbf{d}')$ are even, then $(T_3(\mathbf{d}), T_3(\mathbf{d}')) = 0$, $(\mathbf{d}, \mathbf{d}' \in (\mathbb{F}_4)^n)$.

Proof. By definitions and (7), we get *1*, *2* and *3*. For $\mathbf{d} = (d_1, \cdots, d_n)$, $\mathbf{d}' = (d'_1, \cdots, d'_n) \in (\mathbb{F}_4)^n$, define m_1, m_2, m_3, m_4 by the following:

1. m_1 is the number of i such that i is contained in $\mathrm{supp}(\mathbf{d}) \setminus (\mathrm{supp}(\mathbf{d}) \cap \mathrm{supp}(\mathbf{d}'))$,
2. m_2 is the number of i such that $d_i \neq d'_i$ and that i is contained in $\mathrm{supp}(\mathbf{d}) \cap \mathrm{supp}(\mathbf{d}')$,
3. m_3 is the number of i such that $d_i = d'_i$ and that i is contained in $\mathrm{supp}(\mathbf{d}) \cap \mathrm{supp}(\mathbf{d}')$,
4. m_4 is the number of i such that i is contained in $\mathrm{supp}(\mathbf{d}') \setminus (\mathrm{supp}(\mathbf{d}) \cap \mathrm{supp}(\mathbf{d}'))$.

Then we have $\mathrm{wt}(\mathbf{d}) = m_1 + m_2 + m_3$, $\mathrm{wt}(\mathbf{d}') = m_2 + m_3 + m_4$ and $\mathrm{wt}(\mathbf{d}-\mathbf{d}') = m_1 + m_2 + m_4$. By *3*, we have $(T_3(\mathbf{d}), T_3(\mathbf{d}')) = m_2$. By the assumption, $\mathrm{wt}(\mathbf{d}) \equiv \mathrm{wt}(\mathbf{d}') \equiv \mathrm{wt}(\mathbf{d}-\mathbf{d}') \equiv 0 \pmod 2$; hence we have $m_1 + m_2 + m_3 \equiv m_2 + m_3 + m_4 \equiv m_1 + m_2 + m_4 \equiv 0 \pmod 2$. These equations imply $m_2 \equiv 0 \pmod 2$. Thus $(T_3(\mathbf{d}), T_3(\mathbf{d}')) = m_2 \equiv 0 \pmod 2$. □

Lemma 16 *Assume $q = 2$ or $q = 4$ and $q' = 4$. If $B (\subset (\mathbb{F}_q)^n)$ is self-orthogonal and $D (\subset (\mathbb{F}_4)^n)$ is even, then $C_3(B, D) (\subset (\mathbb{F}_q)^{3\times n})$ is self-orthogonal.*

Proof. Let $\mathbf{b}$, $\mathbf{b}'$ be two elements in B. Since B is self-orthogonal, it follows $(I(\mathbf{b}), I(\mathbf{b}')) = (\mathbf{b}, \mathbf{b}') = 0$. If $\mathbf{d}$, $\mathbf{d}' \in D$ then, by Lemma 15, $(I(\mathbf{b}), T_3(\mathbf{d})) = 0$ and $(T_3(\mathbf{d}), T_3(\mathbf{d}')) = 0$. Therefore $C_3(B, D)$ is self-orthogonal. □

Lemma 17 *Assume that $q = 2$ or $q = 4$ and $q' = 4$ and that n is even. If $B (\subset \mathbb{F}_q^n)$ is self-dual and $D (\subset \mathbb{F}_4^n)$ is an even $[n, n/2]$ code, then $C_3(B, D)$ is a self-dual $[3n, 3n/2]$ code over $\mathbb{F}_q$. In particular, $C_3(B, D)$ is an even code.*

Proof. Since B is self-dual, it follows $\dim_{\mathbb{F}_q}(B) = n/2$. By Theorem 14, we see that $C_3(B, D)$ is a $[3n, 3n/2]$ code over $\mathbb{F}_q$. By the previous lemma, $C_3(B, D)$ is self-orthogonal; hence it is self-dual. □

Theorem 18 *Assume n is even. If B is a binary doubly (resp. singly) even self-dual $[n, n/2, d]$ code and D is an even $[n, n/2, d']$ code over $\mathbb{F}_4$, then $C_3(B, D)$ is a binary doubly (resp. singly) even self-dual $[3n, 3n/2, m]$-code with $m \geq \max\{d, d'\}$.*

Moreover, if

(8) $$\mathrm{supp}(\mathbf{b}) \neq \mathrm{supp}(\mathbf{d})$$

for $\mathbf{b} \in B$ and $\mathrm{d} \in D$ with $\mathrm{wt}(\mathbf{b}) = d$ and $\mathrm{wt}(\mathbf{d}) = d'$, $m > \max\{d, d'\}$.

Proof. By the above Lemmas, it suffices to show that $C_3(B, D)$ is doubly or singly-even. Take any element $x = (x_{ij})$ in $C_3(B, D)$. Recall $K = \{\omega_1 = 1, \omega_2 = \omega, \omega_3 = \bar{\omega}\}$. Define numbers mj $(1 \leq j \leq n)$ by the following:

1. m_1 is the number of columns x_j of x such that one of the three components of x_j is equal to 1,
2. m_2 is the number of columns x_j of x such that two of the three components of x_j is equal to 1,
3. m_3 is the number of columns x_j of x such that all of the three components of x_j are equal to 1.

Then we have $\mathrm{wt}(x) = 3m_3 + 2m_2 + m_1$, $\mathrm{wt}(S(x)) = m_1 + m_3$ and $\mathrm{wt}(L(x)) = m_1 + m_2$. Since $L(x) \in D$ and D is even, it follows $\mathrm{wt}(L(x)) = m_1 + m_2 \equiv 0 \pmod 2$; hence

$$\begin{aligned}\mathrm{wt}(x) &= 3(m_3 + m_1) + 2(m_2 + m_1) - 4m_1 \\ &\equiv 3(m_3 + m_1) \pmod 4.\end{aligned}$$

On the other hand $S(x) \in B$, and if B is doubly-even, then $m_3 + m_1 \equiv 0 \pmod 4$; hence $C_3(B, D)$ is doubly-even. If $\mathbf{b} \in B$, then $I(\mathbf{b}) \in C_3(B, D)$; hence if B is singly-even, then so is $C_3(B, D)$. If $x \in C_3(B, D)$, then $S(x) \in B$, $L(x) \in D$; hence we obtain immediately the assertion for the minimal distance. □

Example 12 (Binary singly-even self-dual [18, 9, 4] code) *Let B be a binary singly-even self-dual code* [6, 3, 2] *and let D be the hexacode. Then $C_3(B, D)$ is a binary singly-even self-dual code* [18, 9, 4].

If B and B' are binary codes with

$$B \cap B' = \langle \mathbf{1} = (1, 1, \cdots, 1) \rangle,$$

then B and $D := B' \otimes \mathbb{F}_4$ satisfy (8). Thus we have the following examples.

Example 13 (Binary Golay [24, 12, 8] code) *Assume e_8 and e'_8 are Hamming* [8, 4, 4] *codes such that* $\dim(e_8 \cap$

$e'_8) = 1$. *Put* $B := e_8$ *and* $D := e'_8 \otimes \mathbb{F}_4$. *Then* $C_3(B, D)$ *is the binary doubly-even self-dual* [24, 12, 8] *code: so it is the binary Golay code. (cf.* Example 1*)*

Example 14 (Binary singly-even self-dual [36, 18, 8] code) *Assume* d_{12} *and* d'_{12} *are binary singly-even self-dual* [12, 6, 4] *codes such that* $\dim(d_{12} \cap d'_{12}) = 1$. *Put* $B := d_{12}$ *and* $D := d'_{12} \otimes \mathbb{F}_4$. *Then* $C_3(B, D)$ *is a binary doubly-even self-dual* [36, 18, 8] *code.*

Example 15 (Binary doubly-even self-dual [72, 36, w] ($w \geq 12$) code) *Assume* $\mathcal{G}$ *and* $\mathcal{G}'$ *are binary Golay* [24, 12, 8] *codes such that* $\dim(\mathcal{G} \cap \mathcal{G}') = 1$. *Put* $B := \mathcal{G}$ *and* $D := \mathcal{G}' \otimes \mathbb{F}_4$. *Then* $C_3(B, D)$ *is a binary doubly-even self-dual* [72, 36, w] ($w \geq 12$) *code.*

It is very interesting to determine the minimal weight w *of this code. Regrettably, we can not determine it.*

Example 16 (Binary doubly-even self-dual [48, 24, 12] code) *Let* B *be a binary* [16, 8, 4] *code generated by the following:*

$b_1 = 1111000000000000$, $b_2 = 0011110000000000$,
$b_3 = 0000111100000000$, $b_4 = 0101010100000000$,
$b_5 = 0000000011110000$, $b_6 = 0000000000111100$,
$b_7 = 0000000000001111$, $b_8 = 0000000001010101$,

and let D *an* $\mathbb{F}_4$*-linear* [16, 8, 6] *code generated by the following:*

$d_1 = 3230000003030020$, $d_2 = 3302020010000010$,
$d_3 = 2301000013200000$, $d_4 = 3300100000120200$,
$d_5 = 2300032000030003$, $d_6 = 3200010010000302$,
$d_7 = 2200001001003003$, $d_8 = 3100000300003130$,

where $2 = \omega$, $3 = \bar{\omega}$. *Then* $C_3(B, D)$ *is a binary doubly-even self-dual* [48, 24, 12] *code.*

It is known that there is only one binary doubly-even self-dual [48, 24, 12]*-code, which is obtained as an extended quadratic residue code* [3].

For convenience, we write the weight distribution and a set of generators of $C_3(B, D)$.

weight	0	12	16	20	24	28	32	36	48
#	1	17296	535095	3995376	7681680	3995376	535095	17296	1

A set of generators for $C_3(B, D)$

coordinate	$((1; 1); (\omega; 1); (\bar{\omega}; 1); (1; 2); (\omega; 2); (\bar{\omega}; 2); \cdots; (\bar{\omega}; 16))$
$T_3(d_1)$	110101110000000000000000001100001100000000101000
$T_3(\omega d_1)$	011110011000000000000000001100001100000000110000
$T_3(d_2)$	110110000101000101000000011000000000000000011000
$T_3(\omega d_2)$	011011000110000110000000101000000000000000101000
$T_3(d_3)$	101110000011000000000000011110101000000000000000
$T_3(\omega d_3)$	110011000101000000000000010101111000000000000000
$T_3(d_4)$	110110000000011000000000000000011101000101000000
$T_3(\omega d_4)$	011011000000101000000000000000010111000011000000
$T_3(d_5)$	101110000000000110101000000000000011000000000110
$T_3(\omega d_5)$	110011000000000011110000000000000011000000000011
$T_3(d_6)$	110101000000000001100000001100000000000011000010 1
$T_3(\omega d_6)$	011110000000000010100000010100000000000001100011 0
$T_3(d_7)$	101101000000000000011000000011000000110000000110
$T_3(\omega d_7)$	110110000000000000010100000010100000001100000000 11
$T_3(d_8)$	110011000000000000000011000000000000011001111000 0
$T_3(\omega d_8)$	011101000000000000000011000000000000011101011000
$I(b_1)$	111111111111000000000000000000000000000000000000
$I(b_2)$	000000111111111111000000000000000000000000000000
$I(b_3)$	000000000000111111111111000000000000000000000000
$I(b_4)$	000111000111000111000111000000000000000000000000
$I(b_5)$	000000000000000000000000111111111111000000000000
$I(b_6)$	000000000000000000000000000000111111111111000000
$I(b_7)$	000000000000000000000000000000000000111111111111
$I(b_8)$	000000000000000000000000111000111000111000111000 111

Finally, we shall study codes $C_3(B, D)$ over $\mathbb{F}_4$.

Lemma 19 *If* B *is an even codes over* $\mathbb{F}_4$ *and* D *is a self-dual code over* $\mathbb{F}_2$, *then* $C_3(B, D \otimes \mathbb{F}_4)$ *is an even code over* $\mathbb{F}_4$.

Proof. Each element $x \in C_3(B, D)$ can be written in the form:

$$x = I(\mathbf{b}) + \sum_{i=1}^{t} \alpha_i\, T_3(\mathbf{d}_i) \qquad (\mathbf{b} \in \mathrm{B}, \mathbf{d}_i \in \mathrm{D} \otimes \mathbb{F}_4, \alpha_i \in \mathbb{F}_4).$$

Put $y := \sum_{i=1}^{t} \alpha_i\, T_3(\mathbf{d}_i)$. Since

$$S(y) = \sum_{i=1}^{t} \alpha_i\, S \circ T_3(\mathbf{d}_i) = 0,$$

it follows that each column of y has one of the following:

$$^t(0, 0, 0),\ ^t(k, 0, k)^{\sigma},\ ^t(1, \omega, \bar{\omega})^{\sigma} \qquad (k \in K = \mathbb{F}_4^{\times}, \sigma \in S_3).$$

Here $^t(a_1, a_2, a_3)^{\sigma} = {}^t(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$. We denote by y_j the j-th column vector of y and define the numbers m_l $(1 \leq l \leq 5)$ by the following:

1. $m_1 = \#\{j \in \mathrm{supp}(\mathbf{b}) \mid y_j = {}^t(1, \omega, \bar{\omega})^{\tau}\}$
2. $m_2 = \#\{j \notin \mathrm{supp}(\mathbf{b}) \mid y_j = {}^t(1, \omega, \bar{\omega})^{\tau}\}$

3. $m_3 = \# \{j \in \mathrm{supp}(\mathbf{b}) \mid y_j = {}^t(x_i, 0, x_i)^\sigma\}$
4. $m_4 = \# \{j \in \mathrm{supp}(\mathbf{b}) \mid y_j = {}^t(k, 0, k)^\sigma, k \neq b_i\}$
5. $m_5 = \# \{j \notin \mathrm{supp}(\mathbf{b}) \mid y_j = {}^t(k, 0, k)^\sigma\}$.

By Lemma 15 *4*,

$$(y, y) = \sum_{i,j=1}^{t} \alpha_i \bar{\alpha}_j (T_3(\mathbf{d}_i), T_3(\mathbf{d}_j)) = 0.$$

On the other hand, since $(y, y) \equiv \mathrm{wt}(y) \pmod 2$, it follows

$$\begin{aligned} \mathrm{wt}(y) &= 3(m_1 + m_2) + 2(m_3 + m_4 + m_5) \\ &\equiv 3(m_1 + m2) \equiv 0 \pmod 2; \end{aligned}$$

hence $m_1 + m_2 \equiv 0 \pmod 2$. Then we have

$$\begin{aligned} \mathrm{wt}(x) &= 3\mathrm{wt}(I(\mathbf{b})) - m_1 - 2m_3 + 3m_2 + 2m_5 \\ &\equiv \mathrm{wt}(\mathbf{b}) \pmod 2. \end{aligned}$$

Since B is even, it follows $\mathrm{wt}(\mathbf{b}) \equiv 0 \pmod 2$; hence $\mathrm{wt}(x) \equiv 0 \pmod 2$. Thus $C_3(B, D)$ is an even code. □

It seems for us that the following holds: If B is an $\mathbb{F}_4$-linear even $[n, n/2, d']$ code. and D is an $\mathbb{F}_4$-linear self-dual $[n, n/2, d]$ code, then $C_3(B, D)$ is an $\mathbb{F}_4$-linear self-dual $[3n, 3n/2, m]$ code, where $m \geq \max\{d, d'\}$.

Example 17 ([6, 3, 4] Hexacode) *Let B be an $\mathbb{F}_4$-linear code* [2, 1, 2] *generated by $b_1 = 1\omega$ and let D* an *$\mathbb{F}_4$-linear code* [2, 1, 2] *generated by $d_1 = 11$. Then $C_3(B, D)$ is the* [6, 3, 4] *Hexacode.*

Example 18 ($\mathbb{F}_4$-linear self-dual [24, 12, 8] code) *Let B be an $\mathbb{F}_4$-linear* [8, 4, 4] *code generated by the following:*

$b_1 = 02010130$, $b_2 = 02001031$,
$b_3 = 00100131$, $b_4 = 12111131$,

where $2 = \omega$ and $3 = \bar{\omega}$ and let D an *$\mathbb{F}_4$-linear* [8, 4, 4] *code generated by*

$d_1 = 11110000$, $d_2 = 00111100$,
$d_3 = 00001111$, $d_4 = 01010101$.

Then $C_3(B, D)$ is an $\mathbb{F}_4$-linear self-dual [24, 12, 8] *code.*

weight	0	8	10	12	14	16	18	20	22	24
#	1	738	12312	177156	1106280	3788217	6206760	4419828	1032408	33516

6. Automorphisms of codes of split type

A code $C(B, D)$ of split type has automorphisms induced by those of B and D. In [6], Griess discusses such automorphisms of the Golay codes. In this section, we shall give a generalization of Griess' argument to codes of split type , however, our notation is somewhat different from Griess'. So we begin with recalling basic facts about automorphisms of codes.

For a field F, we denote by $\mathrm{Mon}(n, F)$ the group of monomial matrices with coefficients in $F^\times$. Let S_n be the symmetric group of degree n. Then S_n acts on the group $(F^\times)^n$ via

$$(a_1, \cdots, a_n) = (a_{1^{\sigma^{-1}}}, \cdots, a_{n^{\sigma^{-1}}}).$$

By this action, we get the semi-direct product $S_n \ltimes (F^\times)^n$. Then the following is an isomorphism :

$$\phi : S_n \ltimes (F^\times)^n \longrightarrow \mathrm{Mon}(n, F), \quad (\sigma, a) \longmapsto A,$$

where

$$a = (a_1, \cdots, a_n), \quad A_{ij} = \begin{cases} a_{i^\sigma} & \text{if } j = i^\sigma \\ 0 & \text{otherwise} \end{cases}$$

For an element $\mathrm{A} \in \mathrm{Mon}(n, F)$, we set

$$\phi^{-1}(A) = (\sigma(\mathrm{A}), a(\mathrm{A})).$$

The monomial group $\mathrm{Mon}(n, F)$ acts on the vector space $W = \sum_{i=1}^n F\mathbf{w}_i$ with basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ via

$$\mathbf{w}_i^{(\sigma, a)} = a_{i^\sigma} \mathbf{w}_{i^\sigma}.$$

The group $\mathrm{Aut}(F)$ acts on the group $\mathrm{Mon}(n, F)$ via

$$(\sigma(A), a(A))^\theta = (\sigma(\mathrm{A}), a(A)^\theta), \quad A \in \mathrm{Mon}(n, F).$$

Set $\mathrm{Mon}^*(n, F) = \mathrm{Aut}(F) \ltimes \mathrm{Mon}(n, F)$, then it acts on the space W via

$$(\sum_{i=1} v_i \mathbf{w}_i)^{(\sigma, A)} = \sum_{i=1}^n v_i^\theta (\mathbf{w}_i)^A = \sum_{i=1}^n (v_{i^{\sigma^{-1}}})^\theta a_i \mathbf{w}_i.$$

Now, we go back to the situation (2) in §2. For a subset $K = \{\omega_i \mid 1 \leq i \leq k\}$ of $\mathbb{F}_{q'}$ with $\sum \omega_i = 0$, let

$$\Omega = \{(c, i) \mid c \in K, 1 \leq j \leq n\}.$$

We consider the set Ω as the standard basis of the space $\mathbb{F}_q^{k \times n}$:

$$V := \mathbb{F}_q^{k\times n} = \sum_{(c,i)\in\Omega} \mathbb{F}_q(c, i).$$

If the set K is $\mathrm{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q)$-invariant, the extended monomial group $\mathrm{Mon}^*(n, \mathbb{F}_{q'})$ acts on Ω via

$$(c, i)^{(\theta, A)} = (c^{\theta} a_{i^{\sigma(A)}}, i^{\sigma(A)}),$$

where

$$(\theta, A) \in \mathrm{Mon}^*(n, \mathbb{F}_{q'}) = \mathrm{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q) \ltimes \mathrm{Mon}(n, \mathbb{F}_{q'}).$$

Hence we have an injection:

(9) $$\mathrm{Mon}^*(n, \mathbb{F}_{q'}) \longrightarrow \mathrm{Mon}(kn, \mathbb{F}_q).$$

Lemma 20 *The linear map* $L : V \longrightarrow \mathbb{F}_{q'}^n$ *is* $\mathrm{Mon}^*(n, \mathbb{F}_{q'})$-*equivariant; i.e.,*

$$L(x^m) = L(x)^m \quad (x \in \mathrm{V}, m \in \mathrm{Mon}^*(n, \mathbb{F}_{q'})).$$

Proof. Set $m = (\theta, A)$, then $m^{-1} = (\theta^{-1}, (A^{\theta-1})^{-1})$. Set

$$x = \sum_{(c,i)\in\Omega} x_{(c,i)}(c, i),$$

then we have

$$\begin{aligned} L(x^m) &= L(\sum_{(c,i)\in\Omega} x_{(c,i)}(c, i)^{(m,0)}) \\ &= L(\sum x_{(c,i)}(c^{\theta} a_{i^{\sigma(m)}}, i^{\sigma(m)})) \\ &= (\cdots, \sum_{c\in K} c^{\theta} a_{i^{\sigma(m)}} x_{(c,i)}, \cdots). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} L(x)^m &= (\cdots, \sum_{c\in K} c x_{(c,i)}, \cdots)^m \\ &= (\cdots, \sum c^{\theta} x_{(c,i)} a_{i^{\sigma(m)}}, \cdots). \end{aligned}$$

Thus we have the equation $L(x^m) = L(x^m)$.

If $K = \mathbb{F}_{q'}$, then $\mathbb{F}_{q'}^n$ acts on Ω via

$$(c, i)^{\mathbf{d}} = (c + d_i, i), \ (\mathbf{d} = (d_1, \cdots, d_n)).$$

Hence the semi-direct product $N := \mathrm{Mon}^*(n, \mathbb{F}_{q'}) \ltimes (\mathbb{F}_{q'})^n$ acts on Ω. Thus we have an injective homomorphism

(10) $$N(= \mathrm{Mon}^*(n, \mathbb{F}_{q'}) \ltimes (\mathbb{F}_{q'})^n) \longrightarrow \mathrm{Mon}(kn, \mathbb{F}_q).$$

Lemma 21 *Assume* $K = \mathbb{F}_{q'}$. *Then the linear map* L *satisfies the following:*

$$L(x^{\mathbf{d}}) = L(x) + (\cdots, (I(\mathbf{e}_i), x)d_i, \cdots).$$

Proof. Let

$$x = \sum_{(c,i)\in\Omega} x_{(c,i)}(c, i) \in V, \mathbf{d} = (d_1, \cdots, d_n) \in (\mathbb{F}_{q'})^n,$$

then we have

$$\begin{aligned} L(x^{\mathbf{d}}) &= L(\sum x_{(c,i)}(c, i)^{\mathbf{d}} \\ &= L(\sum x_{(c,i)}(c + d_i, i)) \\ &= L(\sum x_{(c-d_i,i)}(c, i) \\ &= (\cdots, \sum c x_{(c-d_i,i)}, \cdots) \\ &= (\cdots, \sum (c + d_i) x_{(c,i)}, \cdots) \\ &= L(x) + (\cdots, (I(\mathbf{e}_i), x)d_i, \cdots). \end{aligned}$$

□

Let $(B, D; I, L, T_2)$ be a set of data as in the Case II and $C_2(D)$ the code of split type,i.e., $C_2(D)$ is the linear subspace generated by

$$\{I(\mathbf{e}_1 - \mathbf{e}_j) \mid 1 \le i < j \le n\} \cup \{C_2(\mathbf{d}) \mid \mathbf{d} \in D\}.$$

Recall that $T_2 : \mathbb{F}_{q'}^n \longrightarrow V$ is the section of L defined by

$$T_2(z) = T(z) + I(\mathbf{e}_1) = \sum_{i=1}^{n} (z_i, i) + I(\mathbf{e}_1).$$

The group of extended automorphisms for D is

$$\mathrm{Aut}^*(D) = \{m \in \mathrm{Mon}^*(n, \mathbb{F}_{q'}) \mid D^m = D\}.$$

Then the restriction of the injective homomorphism (10) gives an isomorphism (cf. [6] (5.25),(7.19)):

Proposition 22

$$\mathrm{Aut}^*(D) \ltimes D \simeq \mathrm{Mon}^*(n, \mathbb{F}_{q'}) \ltimes (\mathbb{F}_{q'})^n \cap \mathrm{Aut}(C_2(D)).$$

Proof. If $m \in \mathrm{Aut}^*(D)$, then

$$\begin{aligned} (I(\mathbf{e}_i) - I(\mathbf{e}_j))^m &= I(\mathbf{e}_{i^{\sigma(m)}}) - I(\mathbf{e}_{j^{\sigma(m)}}), \\ (I(\mathbf{e}_1) + T(\mathbf{d}))^m &= I(\mathbf{e}_{1^{\sigma(m)}}) + \mathrm{T}(\mathbf{d}^m). \end{aligned}$$

If $\mathbf{v} \in D$, then

$$(I(\mathbf{e}_i - \mathbf{e}_j))^{\mathbf{v}} = I(\mathbf{e}_i - \mathbf{e}_j),$$

$$(I(\mathbf{e}_1) + T(\mathbf{d}))^{\mathbf{v}} = I(\mathbf{e}_1) + T(\mathbf{d} + \mathbf{v}).$$

Thus, if $(m, \mathbf{v}) \in \mathrm{Aut}^*(D) \ltimes \mathrm{D}$, then $(m, \mathbf{v}) \in \mathrm{Aut}(C_2(D))$.

Conversely, if $(m, \mathbf{v}) \in \mathrm{Aut}(C_2(D))$, then $x^{(m,\mathbf{v})} \in C_2(D)$ for each $x \in C_2(D)$. Set $x = I(\mathbf{e}_1) + T(\mathbf{0}) = T_2(\mathbf{0})$. Then we have $L(x) = 0$, $x^{(m)} = I(\mathbf{e}_{1^{\sigma(m)}}) + T(\mathbf{0})$ and $(I(\mathbf{e}_{1^{\sigma(m)}})) + T(\mathbf{0}), I(\mathbf{e}_i)) = (T(\mathbf{0}), I(\mathbf{e}_i)) = 1$. Therefore

$$\begin{aligned} L(x^{(m,\mathbf{v})}) &= L(x^{(m,\,\mathbf{0}(1,\mathbf{v})}) \\ &= L(x^m) + (\cdots, (I(\mathbf{e}_i), x^m)v_i, \cdots) \\ &= L(x)^m + \mathbf{v} = \mathbf{v}. \end{aligned}$$

Since the left hand side of this is contained in D, it follows $\mathbf{v} \in D$ and $(m, 0) = (m, \mathbf{v})(1, -\mathbf{v}) \in \mathrm{Aut}(C_2(D))$.

Now we set $x = I(\mathbf{e}_1) + T(\mathbf{d})$. Then $L(x^m) = L(I(\mathbf{e}_1) + T(\mathbf{d}))^m = \mathbf{d}^m \in D$. Since $\mathbf{d} \in D$ is arbitrary, m must be in $\mathrm{Aut}^*(D)$. □

Now let $(B, D; I, L, T_3)$ be a set of data as in the Case III with $(q, q') = (q, 4)$. We shall define the subgroup $\mathrm{Aut}(B) \cap \mathrm{Aut}^*(D)$ of $\mathrm{Mon}^*(n, \mathbb{F}_4)$ by

$$\begin{aligned} &\mathrm{Aut}(B) \cap \mathrm{Aut}^*(D) \\ &\quad = \{m \in \mathrm{Mon}^*(n, \mathbb{F}_4) \mid B^{\sigma(m)} = B, D^m = D\}. \end{aligned}$$

By the injection (9), it becomes a subgroup of $\mathrm{Mon}(3n, \mathbb{F}_4)$.

Proposition 23

$$\mathrm{Aut}(B) \cap \mathrm{Aut}^*(D) \simeq \mathrm{Mon}^*(n, \mathbb{F}_4) \cap \mathrm{Aut}(C_3(B, D)).$$

Proof. If $\mathrm{Aut}(B) \cap \mathrm{Aut}^*(D)$, then

$$\begin{aligned} I(\mathbf{b})^m &= I(\mathbf{b}^{\sigma(m)}), \\ T_3(\mathbf{d})^m &= T_3(\mathbf{d}^m), \end{aligned}$$

where $\mathbf{b} \in B$ and $\mathbf{d} \in D$. Since $\sigma(m)$ in $\mathrm{Aut}(B)$ and m in $\mathrm{Aut}^*(D)$, m is contained in $\mathrm{Aut}(C_3(B, D))$.

Conversely, if $m \in \mathrm{Aut}(C_3(B, D))$, then $x^m \in C_3(B, D)$ for each $x \in C_3(B, D)$. For any element $\mathbf{d}$ in D, $x := T_3(\mathbf{d}) \in C_3(B, D)$ and $L(x^m) = L(T_3(\mathbf{d}))^m = \mathbf{d}^m \in \mathrm{D}$; hence $m \in \mathrm{Aut}^*(D)$. Now we set $x = I(\mathbf{b})$, where $\mathbf{b} \in B$. Then $S(x^m) = S(x)^{\sigma(m)} = \mathbf{b}^{\sigma(m)} \in B$; hence $\sigma(m) \in \mathrm{Aut}(B)$. Therefore m is contained in $\mathrm{Aut}(B) \cap \mathrm{Aut}^*(D)$. □

Remark If $B = \mathbf{e}_8$ and $D = \mathbf{e}'_8 \otimes \mathbb{F}_4$, where $\mathbf{e}_8$, $\mathbf{e}'_8$ are Hamming [8, 4, 4]-code with $\dim_{\mathbb{F}_2}(\mathbf{e}_8 \cap \mathbf{e}'_8) = 1$, then $\mathrm{Aut}(B) \cap \mathrm{Aut}^*(D) \simeq S_3 \times L_2(7)$ (cf. Curtis [2], [3]).

Finally we introduce an M-matrix of a code of split type. Let $(B, D; I, L, S)$ be a data as in the situation (2) and C the code of split type associated with it. For each element $(c, i) \in \Omega = \{(c, i) \mid c \in K, 1 \le i \le n\}$, take a non-zero vector $e_{(c,i)}$ in the coordinate line $\mathbb{F}_q(c, i)$. Consider a $k \times n$-matrix

$$M = (f_{(c,i)})$$

obtained by rearranging the set $\{e_{(c,i)}\}$. The matrix M is called an M-matrix of the code C if the linear automorphism f of $V = \sum_{(c,i) \in \Omega} \mathbb{F}_{q'} \simeq (\mathbb{F}_{q'})^{k \times n}$ defined by $(c, i) \longmapsto f_{(c,i)}$ induces an automorphism of the code C. If once we can find methods of making M-matrices, we get many automorphisms of the code C.

Since C is a code of split type, there exist subsets B' and D' of B and D, respectively, such that

$$\{I(\mathbf{b}'), S(\mathbf{d}') \mid \mathbf{b}' \in B', \mathbf{d}' \in D'\}$$

forms a basis of the code C. Therefore M is an M-matrix if the following are contained in C:

$$\{f(I(\mathbf{b}')), f(S(\mathbf{d}')) \mid \mathbf{b}' \in B', \mathbf{d}' \in D'\}.$$

In particular, by Theorem 9, it is easy to check this in the Case II. Thus, for Golay codes, we have nice methods of making M-matrices (cf. Th. 2.5.1 in [12] and Th.4 in [9]). For further discussion and application of M-matrices, we refer to [12], [9], [11], [13], [8] and [10].

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Transport Sector Marginal Abatement Cost Curves in Computable General Equilibrium Model

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Abstract

In the last decade, computable general equilibrium (CGE) models have emerged a standard tool for climate policy evaluation due to their abilities to prospectively elucidate the character and magnitude of the economic impacts of energy and environmental policies. Furthermore, marginal abatement cost (MAC) curves which represent GHG emissions reduction potentials and costs can be derived from these top-down economic models. However, most studies have never address MAC curves for a specific sector that have a large coverage of countries which are needed for allocation of optimal emission reductions. This paper aims to explicitly describe the meaning and character of MAC curves for transport sector in a CGE context through using the AIM/CGE Model developed by Toshihiko Masui. It found that the MAC curves derived in this study are the inverse of the general equilibrium reduction function for CO_2 emissions. Moreover, the transport sector MAC curves for six regions including USA, EU-15, Japan, China, India, and Brazil, derived from this study are compared to the reduction potentials under 100 USD/tCO_2 in 2020 from a bottom-up study. The results showed that the ranking of the regional reduction potentials in transport sector from this study are almost same with the bottom-up study except the ranks of the EU-15 and China. In addition, the range of the reduction potentials from this study is wider and only the USA has higher potentials than those derived from the bottom-up study.

Key Words : Marginal abatement cost curve, Computable general equilibrium model, Top-down approach, Sectoral CO_2 emission, Transport sector

1. Introduction

Recently, the marginal abatement cost (MAC) curves have become an efficient instrument to analyze potentials of GHG mitigation and impacts of the implementation of the Kyoto Protocol and its emission trading[1),2)]. Also, the MAC curves can derive optimal emission reductions for each country which minimizes total abatement cost for a given target[3)]. However, to deal with regionally sector-specific emission reductions, there is no study that provides sectoral MAC curves which have a large coverage of countries and regions yet. For example, Ellerman and Decaux[3)] apply the EPPA Model to generate country-based MAC curves for 12 regions while Sue Wing[4)] develop a multi-sector computable general equilibrium (CGE) model which could generate sectoral MAC curves but only for the United States. As transportation is sharing almost 25% of global CO_2 emissions, it is necessary to treat this sector particularly and analyze its mitigation potentials by sector-based approach. Hence, this paper generates MAC curves for transport sector by region and describes the implication of the sectoral MAC curves in a CGE context through applying the AIM/CGE Model. The algebraic structure and equilibrium of the model are explicitly explained in order to capture the characteristic of the sectoral CO_2 emissions from the transport sector and relevant variables which influence to the MAC curves.

2. Marginal Abatement Cost Curves

There are three ways to represent the abatement costs, i.e., investment cost to implement technological options in order to abate emissions; reductions in GDP due to reduction in production to avoid emissions; and willingness to pay (WTP) to emit more emissions which

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is identical to tax level. WTP is most commonly approximated by the consumer and producer surplus whose consumption and production is affected by the mitigation action[5]. The total abatement cost (TAT) can be measured simply in a framework of partial equilibrium analysis as the net economic loss due to the introduction of CO_2 emission taxation which is so-called deadweight loss (DWL) as shown in Fig. 1. In a general equilibrium model, marginal abatement cost (MAC) is a tax level similarly to the partial equilibrium analysis and the DWL is defined as the reduction in indirect utility divided by marginal utility of income.

In Fig. 1, once a CO_2 tax is levied, consumer surplus will be reduced as the reduction in consumption which results the reduction of CO_2 emissions. Based on this concept, different tax levels will give different reductions in emissions. The coordinates between the CO_2 tax levels and corresponding CO_2 emission reductions can be obtained by varying the levels and then a MAC curve can be plotted as shown in Fig. 2. As this MAC curve derived from the economic impact, the area under the MAC represents the total abatement cost which equals to the DWL.

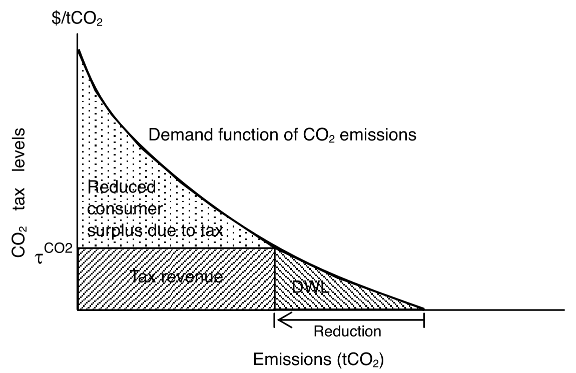


Fig. 1. Economic impact due to CO_2 tax

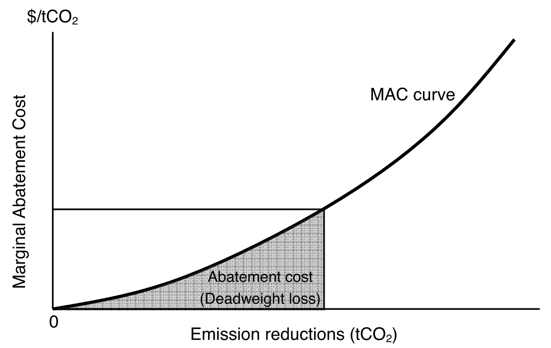


Fig. 2. The marginal abatement cost curve

There are two approaches to generate MAC curves; "bottom-up" engineering/technology-based models and "top-down" economic models[1),6),7)]. The first approach simulates the interactions among the technologies that form the economy's energy system. The bottom-up models contain detailed empirical information on the technical characteristics of specific abatement options. It means that the bottom-up approach represents the direct cost of the available abatement technologies. On the other hand, the top-down models are based on aggregated microeconomic models. The models are most often computable general equilibrium (CGE) models which contain the information of abatement technologies only implicitly. The top-down models treat abatement costs purely as the profit or utility foregone as a result of forced changes in behavior induced by environmental policy[8)]. As the estimation of MAC is essential to assess the potential of climate change mitigation, the cost estimation studies for countries through both top-down and bottom-up approaches have been discussed extensively in the Assessment Reports of IPCC Working Group III since the Second Assessment Report (SAR). The costs estimated by the bottom-up studies that rely on more detailed and comprehensive assessments of technological options tended to arrive at larger efficiency potentials and lower costs of saved energy than the less detailed studies. The comparison of top-down model and bottom-up modeling methodologies has been discussed in the IPCC Third Assessment Report (TAR). However, the comparison of GHG mitigation potentials by country had been done only within the same approach. Latter, the comparison of sectoral potentials for the global GHG mitigation estimated by bottom-up and top-down approaches had been made in the IPCC Forth Assessment Report (AR4). Surprisingly, several sectors by the top-down models, for example, energy supply, buildings, and industry sectors indicate a higher emission reduction than the bottom-up approaches. One of the reasons is noted that top-down models allow for product substitution, which is often excluded in bottom-up sector analysis. Also, it found that the differences between bottom-up and top-down are larger at the sector level. The existing studies however have not been compared the potentials of the transport sector by country between the bottom-up and top-down models which is one of objectives of this study.

Based on the literature review, it is practical difficulty to develop the MAC curves for transport sector which have a large coverage of countries and options to meeting the objective of this study. Also, we aim at assessment of reduction potentials and abatement costs of CO_2 emissions across sector in general—not specific abatement technologies. Therefore, in this study we employ a multi-region multi-sector CGE model which could tackle GHG emissions from a specific sector covering major emitting countries and regions. In a CGE model, marginal abatement cost curves can be derived when the costs associated with different levels of reductions or the reduction targets associated with different abatement costs are generated which will be further explained in next sections.

3. Modeling Sectoral CO^2 Emissions in a CGE Context

In a CGE model, CO_2 emissions are primarily associated with the use of fossil fuels (i.e. coal, oil and gas) as intermediate inputs to production sectors and as final consumption demand to household as shown in Fig. 3. The main actors in the diagram are households, who own primary factors of production (e.g. capital, labor and natural resources) and the final consumers of produced commodities, and firms, who rent the factors of production from the households for the purpose of producing single goods and services that the household then consume. The critical data that determine the structure of a CGE model are contained in social accounting matrix (SAM), which represents a snapshot of the economy of each region[9].

Each production sector produces single commodity or service by inputting intermediate goods and primary factors. To address energy and climate policies, intermediate inputs for production and produced goods for final consumption are divided into non-energy and energy goods. Some production sectors of non-energy goods/services use a relatively large proportion of energy goods (i.e. fossil fuels and electricity) as inputs, such as energy intensive productions, metal and machinery, and transport. Energy goods include fossil fuels which are carbon content goods, and electricity. Then, each fossil fuel (i.e. coal, oil and gas) is modeled as a composite with carbon emissions by a Leontief form, i.e. the elasticity of substitution equals zero. These fossil fuels composites are crucially important that we can deal with CO_2 emission tax by introducing price of CO_2 emission permits. Similar to production sectors, we can track fossil fuels consumption and its CO_2 emissions in final consumption sector as shown in the diagram.

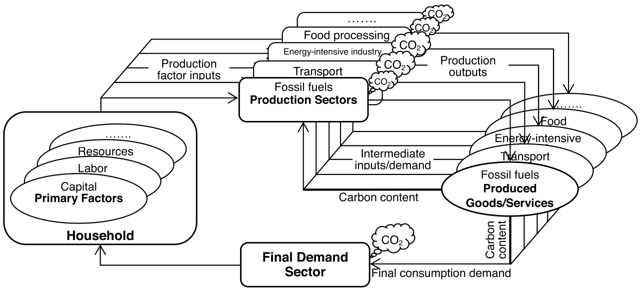


Fig. 3. A multi-sector CGE framework with CO_2 emission

4. The AIM/CGE Model

4.1 Overview

In this study, we employed a global CGE model namely the AIM/CGE Model developed by Masui[10]. The AIM stands for Asia-Pacific Integrated Model which is a large-scale computer simulation model of the National Institute for Environmental Studies (NIES), aiming to assess the climate change problem[11),12)]. The AIM/CGE Global Model is written by the GAMS/MPSGE modeling system[13)], based on GTAPinGAMS and GTAP-EG datasets[14)]. The global economic data used in the model is based on GTAP version 6 which has base year of the data in 2001 and disaggregates the global economy into 87 regions and 57 sectors. Nevertheless, the AIM/CGE model was added many items, for example, more GHGs, biomass, and power generation technologies. The model aggregates the GTAP dataset into 24 regions, 22 production sectors and a final consumption sector[15)], as presented in Table 1. The AIM/CGE model has dynamic structure which can simulate the global economy in the base year, 2001 and from 2010 to 2110 with a 10-year span. The target year of this study is 2020. The study aims to develop MAC curves for the transport sector in 2020 and then utilize the derived MAC curves to analyze CO_2 emission reduction potential in the transport sector by region for the post-Kyoto Protocol which will be discussed in the section 6.

Table 1 Regions and sectors in the AIM/CGE Global Model

Countries and Regions	Production Sectors
Developed Countries	***Non-Energy***
Japan (JPN)	1. Food (FOD)
Australia (AUS)	2. Energy intensive products (EIS)
New Zealand (NZL)	3. Metal and machinery (M_M)
Canada (CAN)	4. Other manufactures (OMF)
United States of America (USA)	5. Water (WTR)
Russia (RUS)	6. Construction (CNS)
Western Europe (EU15)	*7. Transport (TRT)*
Eastern Europe (EU10)	8. Communication (CMN)
Rest of Europe (XRE)	9. Public service (OSG)
Developing Countries	10. Other service (SER)
Korea (KOR)	11. Investment (CGD)
China (CHN)	12. Agriculture (AGR)
Indonesia (IDN)	13. Livestock (LVR)
India (IND)	14. Forestry (FRS)
Thailand (THA)	15. Fishing (FSH)
Other South-east Asia (XSE)	16. Mining, except fossil fuels (OMN)
Other South Asia (XSA)	***Energy***
Rest of Asia-Pacific (XRA)	1. Coal (COA)
Mexico (MEX)	2. Crude oil (OIL)
Argentine (ARG)	3. Petroleum products (P_C)
Brazil (BRA)	4. Gas (GAS)
Other Latin America (XLM)	5. Gas manufacture distribution (GDT)
Middle East (XME)	6. Electricity (ELY)
South Africa (ZAF)	**Households**
Other Africa (XAF)	Final consumption

4.2 The structure of the model

All production and final consumption sectors are modeled using nested Constant Elasticity of Substitution (CES) production functions, or Cobb-Douglas (C-D) and Leontief (LT) forms, which are special case of the CES, as shown in Figs. 4 and 5. Typical productions of non-energy sectors (including the transport sector) have the structure as shown in Fig. 4. At the top of the production tree, each sector i in a region r produces a composite commodity that can be sold domestically or exported to other regions[16]. The relationship between domestic and export goods can be represented by a Constant Elasticity of Transformation (CET) function as

$$y_{i,r} = \gamma_{i,r}\left[\beta_{i,r}^{Y} D_{i,r}^{\frac{\sigma_i^Y+1}{\sigma_i^Y}} + (1-\beta_{i,r}^{Y}) E_{i,r}^{\frac{\sigma_i^Y+1}{\sigma_i^Y}}\right]^{\frac{\sigma_i^Y}{\sigma_i^Y+1}} \qquad (1)$$

where, $y_{i,r}$ is sector i's total output, $\gamma_{i,r}$ is the output efficiency parameter, $\beta_{i,r}^{Y}$ is the share parameter, $D_{i,r}$ represents sector i's supply for domestic, $E_{i,r}$ is the sector's output supply for export, and σ_i^x is the CET for sector i. Each firm allocates it's output between domestic and export markets to maximize revenue, subject to the CET function, yielding export goods output per unit of domestic goods output as a function of relative prices,

$$\frac{E_{i,r}}{D_{i,r}} = \left[\left(\frac{1-\beta_{i,r}^{Y}}{\beta_{i,r}^{Y}}\right)\left(\frac{p_{i,r}^{D}}{p_{i,r}^{E}}\right)\right]^{-\sigma_i^Y} \qquad (2)$$

where, $p_{i,r}^{D}$ and $p_{i,r}^{E}$ are, respectively, prices of domestic and exported commodities from sector i.

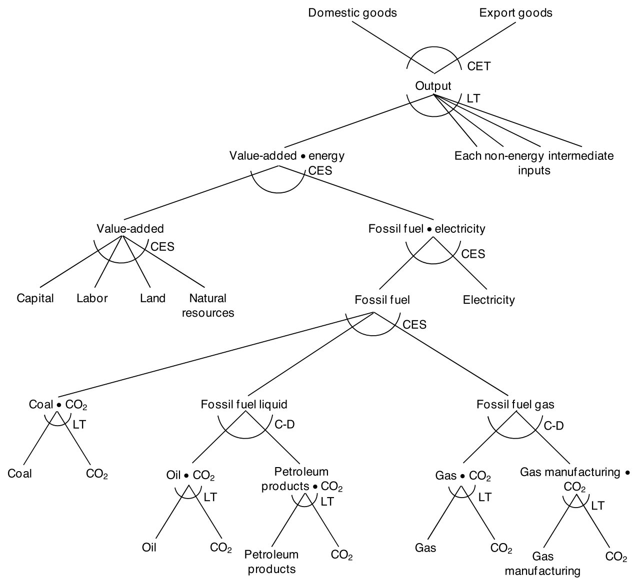


Fig. 4. Production structure (non-energy sectors)

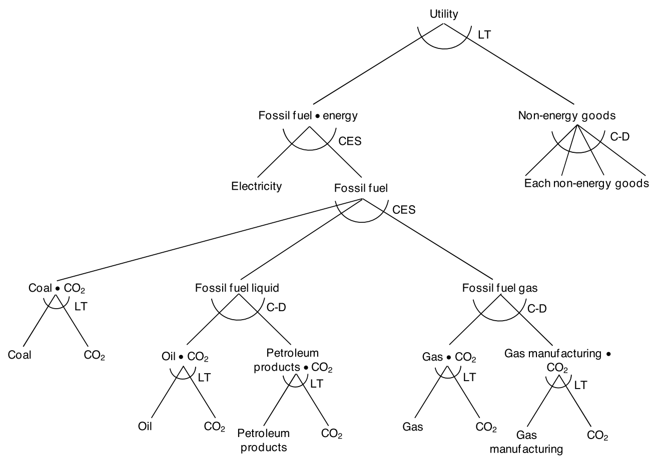


Fig. 5. Final consumption

The composite output above is produced with fixed-coefficient (Leontief) inputs of each non-energy intermediate goods and an energy-primary factor composite. The energy-primary factor composite is a CES function. Primary factor (i.e. value-added) inputs of capital, labor, land and natural resources are aggregated through a CES production function. The energy composite is a CES function of electricity and fossil fuels. At each node of the production tree, industries will decide on volume of each input in order to minimize production cost. The producer behavior is be formulated as shown in Appendix A. Fossil fuel production has a different structure—its output is produced as an aggregate of a resource input and a non-resource input composite. Final demand has the structure shown in Fig. 3. Utility in each region is a Leontief aggregate of energy and non-energy goods. The household behavior is formulated as shown in Appendix B. Main parameters influence to demand and substitutability both production and consumption functions given in Appendixes A and B are share parameters (or input coefficients in case of Leontief form) and elasticity of substitution, respectively. The share parameter for each input both single and composite input can be calibrated by using the benchmark data from the GTAP database. For example, input coefficient for each non-energy goods ($a_{ne,j}$) and input coefficient for value-added and energy goods composite ($a_{ave,j}$) are involved to determine the output volume of production sector. The elasticity of substation is also derived from the GTAP database. The elasticity of substitution, for example, at the 2nd level of aggregation of value-added composite and fossil fuel-electricity composite (σ_{vae}) is involved in aggregation along with share parameters of value-added composite and fossil fuel-electricity composite to determine the input volume of value-added and energy goods composite for the top level of production.

Intermediate inputs for productions and final demand for consumption are generated through the Armington aggregation[17] which mixes domestic and imported goods as imperfect substitutes, specified as a CES function as shown in Fig. 6. The CES function representing the relationship between the two categories of intermediate inputs can be expressed as

$$x_{i,j,r} = \vartheta_{i,j,r}\left[\beta^X_{i,j,r} D_{i,j,r}^{\frac{\sigma_i^X-1}{\sigma_i^X}} + (1-\beta^X_{i,j,r}) M_{i,j,r}^{\frac{\sigma_i^X-1}{\sigma_i^X}}\right]^{\frac{\sigma_i^X}{\sigma_i^X-1}} \tag{3}$$

where, $x_{i,j,r}$ is composite intermediate goods from sector i to sector j, $\vartheta_{i,j,r}$ is the intermediate input efficiency parameter, $\beta^X_{i,j,r}$ is the share parameter, $D_{i,j,r}$ represents domestic intermediate goods, $M_{i,j,r}$ represents imported intermediate goods, and σ_i^X is the CES for sector i.

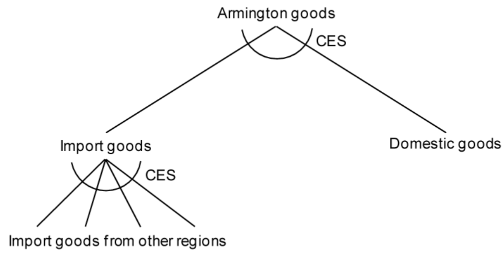


Fig. 6. International trade

Each firm decides on inputting volume between the two sources of intermediate inputs to minimize cost, subject to the CES function, yielding import demand per unit of domestic demand as a function of relative prices,

$$\frac{M_{i,j,r}}{D_{i,j,r}} = \left[\left(\frac{1-\beta^X_{i,j,r}}{\beta^X_{i,j,r}}\right)\left(\frac{p^D_{i,r}}{p^M_{i,r}}\right)\right]^{\sigma_i^X} \tag{4}$$

where, $p^D_{i,r}$ and $p^M_{i,r}$ are, respectively, prices of domestic and exported commodities from sector i.

The AIM/CGE Model represents the government passively that collect taxes and disburses the revenues to households as lump-sum transfers. Saving and investment by sector in a region is modeled endogenously through the sector 11 that collect produced goods from other sectors to investment.

4.3 The equilibrium conditions with CO_2 emission constraint

At equilibrium, the model will solve for the set of commodity and factor prices, and the levels of industry activity and household income that clear all markets in the economy, given aggregate factor endowments, households' consumption technologies and industries' transformation technologies. Profit maximization in the constant-returns-to-scale case implies that no activity earns a positive profit. On the consumer's side, in equilibrium income restricts expenditure, i.e. there is no excess demand of the household, including government. Such equilibrium can be represented by the three conditions; (1) zero profit (2) market clearance and (3) income balance conditions as shown in Appendix C. As the main parameters; share parameters and elasticity of substitution, reflect demand and substitutability, all equilibrium conditions involve these parameters as well. For example, the zero profit conditions, price of output ($py_{j,r}$) and expenditure index (θ_r) are function of all input prices at the domestic market which involves both share parameters and elasticity of substitution.

5. Implication of Sectoral MAC Curve in the AIM/CGE Model

As mentioned that CO_2 emissions emitted from sectors in a CGE model can be determined through intermediate inputs of fossil fuels into that sector with emission factor of each fossil fuel. In the benchmark data (i.e. base case), the CO_2 emissions tax is equal to zero, consequently production sectors and household will input and consume fossil fuel regardless amount of CO_2 emitted. Once we introduce CO_2 emission tax (or price of emission permit), the price of consuming fossil fuel will increase. CO_2 emissions by sector *j* ($Q_{j,r}^{CO2}$) and final consumption ($Q_{c,r}^{CO2}$) in region *r* then can be calculated by eq. (5) and (6), respectively.

$$Q_{j,r}^{CO2} = \sum_{i=17}^{21} \emptyset_i x_{i,j,r} \tag{5}$$

$$Q_{c,r}^{CO2} = \sum_{i=17}^{21} \emptyset_i c_{i,r} \tag{6}$$

where, $x_{i,j}$ and $x_{i,c}$ = inputting and consuming volume of fossil fuel (sectors 17 to 21 are, respectively, coal, crude oil, petroleum products, gas, and gas manufacture distribution) into sector *j* and final consumption, respectively, which can be determined as shown in Appendixes A and B.

This paper specially focuses on MAC curves for transport sector which shares around 25% of global CO_2 emissions, mainly from fossil fuel combustion. At the equilibrium, CO_2 emissions from the transport sector (sector no. 7) for region *r* can be determined by eq. (7). All composite prices in eq. (7) can be further determined by eq. (8) to (13). All variables and parameters of the equations are defined in Appendix A.

$$\begin{aligned} Q_{7,r}^{CO_2} = &\left\{ \frac{\emptyset_{17}\alpha_{17,7,r}^{\sigma_{ff}}}{(p_{17,r}+\tau^{CO2}\emptyset_{17})^{\sigma_{ff}}} + \frac{\emptyset_{18}\alpha_{18,7,r}p_{lq,7,r}^{1-\sigma_{ff}}}{p_{18}+\tau^{CO2}\emptyset_{18}} + \frac{\emptyset_{19}\alpha_{19,7,r}p_{lq,7,r}^{1-\sigma_{ff}}}{p_{19}+\tau^{CO2}\emptyset_{19}} + \frac{\emptyset_{20}\alpha_{20,7,r}p_{gs,7,r}^{1-\sigma_{ff}}}{p_{20}+\tau^{CO2}\emptyset_{20}} \right. \\ &\left. + \frac{\emptyset_{21}\alpha_{21,7,r}p_{gs,7,r}^{1-\sigma_{ff}}}{p_{21}+\tau^{CO2}\emptyset_{21}} \right\} \alpha_{ff,7,r}^{\sigma_{fe}}\alpha_{fe,7,r}^{\sigma_{vae}} a_{vae,7,r} p_{ff,7,r}^{\sigma_{ff}-\sigma_{fe}} p_{fe,7,r}^{\sigma_{fe}-\sigma_{vae}} p_{vae,7,r}^{\sigma_{vae}} y_{7,r} \end{aligned} \tag{7}$$

$$p_{lq,7,r} = \frac{1}{\beta_{lq,7,r}} \left(\frac{p_{18,r}+\tau^{CO2}\emptyset_{18}}{\alpha_{18,7,r}}\right)^{\alpha_{18,7,r}} \left(\frac{p_{19,r}+\tau^{CO2}\emptyset_{19}}{\alpha_{19,7,r}}\right)^{\alpha_{19,7,r}} \tag{8}$$

$$p_{gs,7,r} = \frac{1}{\beta_{gs,7,r}} \left(\frac{p_{20,r}+\tau^{CO2}\emptyset_{20}}{\alpha_{20,7,r}}\right)^{\alpha_{20,7,r}} \left(\frac{p_{21,r}+\tau^{CO2}\emptyset_{21}}{\alpha_{21,7,r}}\right)^{\alpha_{21,7,r}} \tag{9}$$

$$p_{ff,7,r} = \left[\alpha_{17,7,r}^{\sigma_{ff}}\left(p_{17,r}+\tau^{CO2}\emptyset_{17}\right)^{1-\sigma_{ff}} + \alpha_{lq,7,r}^{\sigma_{ff}}p_{lq,7,r}^{1-\sigma_{ff}} + \alpha_{gs,7,r}^{\sigma_{ff}}p_{gs,7,r}^{1-\sigma_{ff}}\right]^{\frac{1}{1-\sigma_{ff}}} \tag{10}$$

$$p_{fe,7,r} = \left(\alpha_{ff,7,r}^{\sigma_{fe}}p_{ff,7,r}^{1-\sigma_{fe}} + \alpha_{22,7,r}^{\sigma_{fe}}p_{22,r}^{1-\sigma_{fe}}\right)^{\frac{1}{1-\sigma_{fe}}} \tag{11}$$

$$p_{vae,7,r}^{\sigma_{vae}} = \left(\alpha_{va,7,r}^{\sigma_{vae}}p_{va,7,r}^{1-\sigma_{vae}} + \alpha_{fe,7,r}^{\sigma_{vae}}p_{fe,7,r}^{1-\sigma_{vae}}\right)^{\frac{1}{1-\sigma_{vae}}} \tag{12}$$

$$p_{va,7,r} = \left(\alpha_{k,7,r}^{\sigma_{va}}K_{7,r}^{\frac{\sigma_{va}-1}{\sigma_{va}}} + \alpha_{w,j}^{\sigma_{va}}W_{7,r}^{\frac{\sigma_{va}-1}{\sigma_{va}}} + \alpha_{l,7,r}^{\sigma_{va}}L_{7,r}^{\frac{\sigma_{va}-1}{\sigma_{va}}} + \alpha_{r,7,r}^{\sigma_{va}}R_{7,r}^{\frac{\sigma_{va}-1}{\sigma_{va}}}\right)^{\frac{\sigma_{va}}{\sigma_{va}-1}} \tag{13}$$

From the equations above, we can see that the emission of the sector is a function of emission factors ($\emptyset$), CO_2 emission tax (τ^{CO2}), production factors, all prices related to fossil fuel, and the sector's output ($y_{7,r}$). Also, it can be said that the emission of a sector is proportional to its output which can be obtained at market equilibrium that depends on all prices and CO_2 tax.

Fig. 5 shows conceptually the relationship between the general equilibrium demand function (thick curve) and demand functions (thinner curve) for CO_2 emissions. The demand function is the emission level at a certain level of CO_2 tax given the price of other goods and inputs at a certain value. It will shift corresponding to the change in given price level of other goods as shown by the dash curves in Fig. 7. On the other hand the general equilibrium demand function expresses the emission level at a certain level of CO_2 tax given the price of other goods and inputs at the equilibrium level corresponding to the given CO_2 tax level. At a certain level of CO_2 tax (τ^{CO2}), therefore, we can determine the crossing point between two curves where all price levels are at the equilibrium level corresponding to the given CO_2 tax level. The locus of crossing points is the general equilibrium demand function.

The general equilibrium emission reduction function is defined as the difference in the emission quantity of the given level of CO_2 tax from the emission level of no CO_2 tax case. The marginal abatement cost (MAC) curves is defined as the inverse of general equilibrium emission reduction function with respect to the tax level of CO_2 which is shown as the CO_2 tax level for a given level of reduction of the general equilibrium emission reduction function. The total abatement cost is equal to the area under the MAC curve, which is identical to the deadweight loss (DWL) for a given CO_2 tax as mentioned in Section 1 and shown in Fig. 2.

6. Comparing the Transport MAC Curves to the Bottom-Up Approach

In practical, to generate sectoral MAC curves, we imposed CO_2 taxes into the model and varied from 0 up to 200 USD/tCO_2 and obtained corresponding CO_2 emissions by sector by region. With having the

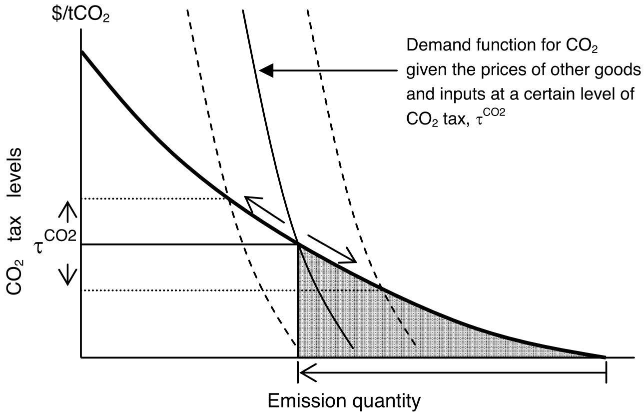
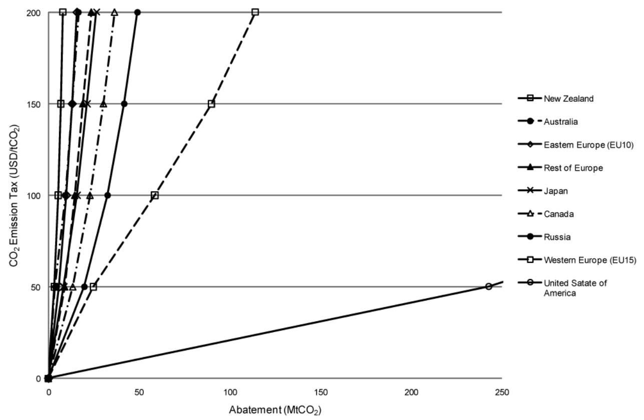


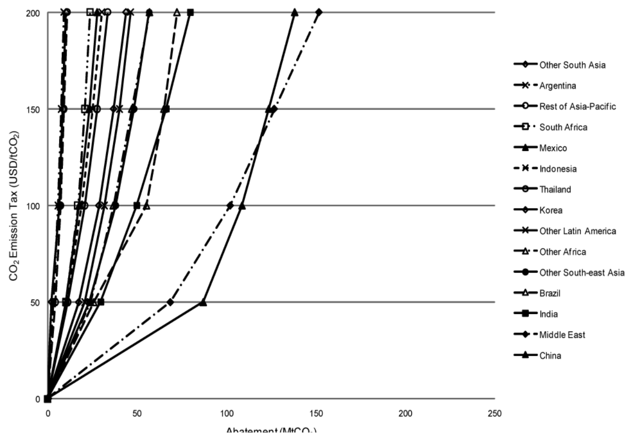
Fig. 7. The general equilibrium demand function and demand functions for CO_2 emissions

coordinates of CO_2 emission taxes and associated emission reductions, we can plot sectoral MAC curves by region. Fig. 8 shows the derived MAC curves for transport sector by region in 2020. It can be interpreted straight forward that USA has high potential of CO_2 emission reductions in transport sector, i.e. abatement cost of CO_2 emissions is cheapest and much cheaper than other countries (see Fig. 8 (a)). For developing countries, abatement cost of CO_2 emissions in transport sector are also cheap; particularly, China, India, Brazil and a group of Middle-East countries (see Fig. 8 (b)). A major reason of why the effects of the CO_2 emission taxes are particularly strong in the USA but are very weak in the other developed countries is that the fossil fuel prices and taxes in the USA are very lower than other countries. From key world energy statistics published by the International Energy Agency[18)], gasoline price in the USA is cheaper than other countries, e.g. gasoline price in Japan is more than two times of the USA price. Thus, when we introduce a CO_2 emission tax into the model, reductions in fossil fuel use in the USA are very sensitive. As the technology (i.e. represented by production function) of the transport sector, specifically the substitution rate between capital and energy for the USA and Japan are similar, then the price level of fossil fuels could be the reason for the difference of the sensitivity to the CO_2 emission taxes between the USA and Japan. For Japan, fossil fuel taxes are relatively high. With the same level of the CO_2 emission tax with the USA, reductions in fossil fuel use in Japan are very small. Also, energy efficiencies in Japan, particularly in the transport sector, are considerably high. It will be very expensive to reduce more a unit of CO_2 emissions in the transport sector for Japan. This is similar for other developed countries like the EU-15, Australia and New Zealand.

The MAC curves derived from this study are then compared to the GHG mitigation potentials based on the bottom-up approach by the NIES[19]. The CO_2 reduction potentials in transport sector under 100 USD/ tCO_2 for six regions from the bottom-up study are roughly read from the graphic. This comparison aims to assess consistency of the MAC curves derived from the different approaches only. Table 2 shows the result of the comparison between the bottom-up model and this study. The result showed that the ordering of the regional reduction potentials from both studies is almost same (e.g., the USA is the cheapest countries while Japan is most expensive), expect for the ranks of the EU-15 and China. The range of the reduction potentials



(a) Developed Countries



(b) Developing Countries

Fig. 8. The derived MAC curves for transport sector by region in 2020

Table 2 Comparison of CO_2 emission reduction potentials in transport sector

Country	Reduction Potentials ($MtCO_2$) under 100 USD/tCO_2		%
	Bottom-up Approach[19]	This Study	difference
USA	234.0	399.4	+70%
EU-15	177.3	58.6	-67%
China	134.8	108.7	-19%
Brazil	67.6	55.4	-18%
India	61.5	49.7	-19%
Japan	56.7	15.8	-72%

from this study is wider (i.e., 15.8 – 399.4 $MtCO_2$), and only the reduction potential of USA from is higher than the bottom-up study. Also, the reduction potentials of the EU-15 and Japan from this study are completely lower than the bottom-up study (about 70% below). The reduction potentials of developing countries including China, Brazil and India from this study are quite same as the bottom-up model's results (about 20% below). In addition, based on the results from this study, we found that the abatement costs of the top-down study is not necessary larger than the bottom-up study as noted by previous studies[6),7)].

To clarify relative price effect and demand effect, a relationship between fuel price with CO_2 tax and fuel demand by the transport sector for the countries which have big difference; USA, EU-15 and Japan, is shown in Fig. 9. It shows that the influence of fuel price with CO_2 tax to the transportation fuel demand is very strong in case of USA and relatively weak in case of EU-15 and Japan. Therefore, the price elasticity could be one of the reasons why the reduction potential of USA is totally opposite to those of EU-15 and Japan when compared to the reduction potential by the bottom-up approach.

From the comparison analysis in this study, there are main reasons behind the differences between bottom-up and top-down modeling results. As mentioned that fuel price and tax in the EU-15 and Japan are very high comparing to USA. The change in price of fuel due to CO_2 tax introduced will influence to change in demand of transportation fuel slightly. This is opposite for the USA's case. Therefore, the price elasticity would be reasons of the differences between the bottom-up and top-down models, particularly, for the USA, EU-15 and Japan. Another reason would be elasticity of substitution in the top-down model which is often

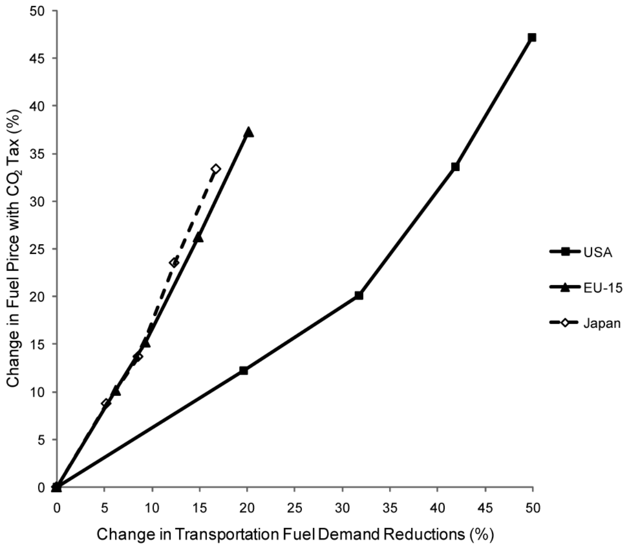


Fig. 9. Relationship between fuel price with CO_2 tax and the transportation fuel demand by country

excluded in the bottom-up sector analysis. This leads the range of the reduction potentials by sector estimated by the top-down model wider than those from the bottom-up model. Therefore, it can be concluded that the differences between bottom-up and top-down approaches are larger when considering at the sector level as same as mentioned in the IPCC's AR4.

7. Concluding Remarks

In this paper, we described explicitly the equations to derive MAC curves for transport sector by region through using the multi-region multi-sector CGE model. It found that the sectoral MAC curves derived from this paper are the inverse of the general equilibrium reduction function for CO_2 due to that it depends on all prices in the economy including the emission tax given. Moreover, the derived MAC curves for transport sector for regions are compared to the GHG mitigation potentials derived from a bottom-up study. At the CO_2 tax level of 100 USD/tCO_2, the ranking of the regional reduction potentials in transport sector in 2020 from this study are almost same with the bottom-up study, except the orders of the EU-15 and China. The reduction potentials from the top-down model are considerably lower than the bottom-up model expect for the USA. Nevertheless, the both studies also showed that the USA is the cheapest countries and Japan is most expensive to reduce CO_2 emissions in transport sector. The main factors which influence the differences between bottom-up and top-down modeling results by sector that could be concluded by this study are the price elasticity and elasticity of substitution which are relatively different among sectors. It would be also the reason of why the reduction potentials by the top-down model are wider than the bottom-up model. This conclusion supports the issue noted in the IPCC's AR4 that the differences between bottom-up and top-down approaches are larger at the sector level.

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Appendix A: Formulation of producers' behaviors

Level	Cost minimization	Demand function
1	$C_j = \min \sum_{ne=1}^{16} p_{ne} x_{ne,j} + p_{vae,j} VAE_j$ s.t. $y_j = \min \left\{ \frac{x_{1,j}}{a_{1,j}}, \ldots, \frac{x_{ne,j}}{a_{ne,j}}, \ldots, \frac{x_{16,j}}{a_{16,j}}, \frac{VAE_j}{a_{vae,j}} \right\}$	$x_{ne,j} = a_{ne,j} y_j \qquad VAE_j = a_{ave,j} y_j$ $C_j = p_j y_j = \left(a_{vae,j} p_{vae,j} + \sum_{ne=1}^{16} a_{ne,j} p_{ne} \right) y_j$
	C_j: production cost, $x_{ne,j}$: intermediate goods input from non-energy sector *ne* to industry j, VAE_j : input volume of value-added and energy, p_{ne}: price of non-energy commodity ne, $p_{vae,j}$: unit cost of value-added and energy input, $a_{ne,j}, a_{vae,j}$: input coefficient, p_j: price of commodity j, y_j : output volume	
2	Value-added–energy (VAE) composite $C_j^{vae} = \min\ p_{va,j} VA_j + p_{fe,j} FE_j$ s.t. $VAE_{j,r} = \left(\alpha_{va,j} VA_j^{\frac{\sigma_{vae}-1}{\sigma_{vae}}} + \alpha_{fe,j} FE_j^{\frac{\sigma_{vae}-1}{\sigma_{vae}}} \right)^{\frac{\sigma_{vae}}{\sigma_{vae}-1}}$	$VA_j = \frac{\alpha_{va,j}^{\sigma_{vae}} p_{vae,j}^{\sigma_{vae}}}{p_{va,j}^{\sigma_{vae}}} VAE_j \qquad FE_j = \frac{\alpha_{fe,j}^{\sigma_{vae}} p_{vae,j}^{\sigma_{vae}}}{p_{fe,j}^{\sigma_{vae}}} VAE_j$ $C_j^{vae} = p_{vae,j} VAE_{j,r}$ $= \left[\left(\alpha_{va,j}^{\sigma_{vae}} p_{va,j}^{1-\sigma_{vae}} + \alpha_{fe,j}^{\sigma_{vae}} p_{fe,j}^{1-\sigma_{vae}} \right)^{\frac{1}{1-\sigma_{vae}}} \right] VAE_{j,r}$
	C_j^{vae}: value-added–energy cost, VA_j : input volume of value-added, FE_j : input volume of fossil fuel–energy, $p_{va,j}, p_{fe,j}$: unit cost of value-added and fossil fuel–energy, $\alpha_{va,j}$, $\alpha_{fe,j}$: parameters, σ_{vae}: elasticity of substitution	
3	Value-added (VA) composite $C_j^{va} = \min\ p_k K_j + p_w W_j + p_l L_j + p_r R_j$ s.t. $VA_j = \left(\alpha_{k,j} K_j^{\frac{\sigma_{va}-1}{\sigma_{va}}} + \alpha_{w,j} W_j^{\frac{\sigma_{va}-1}{\sigma_{va}}} + \alpha_{l,j} L_j^{\frac{\sigma_{va}-1}{\sigma_{va}}} + \alpha_{r,j} R_j^{\frac{\sigma_{va}-1}{\sigma_{va}}} \right)^{\frac{\sigma_{va}}{\sigma_{va}-1}}$	$K_j = \frac{\alpha_{k,j}^{\sigma_{va}} p_{va,j}^{\sigma_{va}}}{p_k^{\sigma_{va}}} VA_j \quad W_j = \frac{\alpha_{w,j}^{\sigma_{va}} p_{va,j}^{\sigma_{va}}}{p_w^{\sigma_{va}}} VA_j \quad L_j = \frac{\alpha_{l,j}^{\sigma_{va}} p_{va,j}^{\sigma_{va}}}{p_l^{\sigma_{va}}} VA_j$ $R_j = \frac{\alpha_{r,j}^{\sigma_{va}} p_{va,j}^{\sigma_{va}}}{p_r^{\sigma_{va}}} VA_j \qquad C_j^{va} = p_{va,j} VA_j = \left[\left(\alpha_{k,j}^{\sigma_{va}} K_j^{\frac{\sigma_{va}-1}{\sigma_{va}}} + \alpha_{w,j}^{\sigma_{va}} W_j^{\frac{\sigma_{va}-1}{\sigma_{va}}} + \alpha_{l,j}^{\sigma_{va}} L_j^{\frac{\sigma_{va}-1}{\sigma_{va}}} + \alpha_{r,j}^{\sigma_{va}} R_j^{\frac{\sigma_{va}-1}{\sigma_{va}}} \right)^{\frac{\sigma_{va}}{\sigma_{va}-1}} \right] VA_j$
	C_j^{va}: value-added cost, K_j , W_j , L_j , R_j : capital, labor, land, and natural resource input volume, p_k , p_w, p_l , p_r : unit rents for capital, labor, land, and natural resource, $\alpha_{k,j}$, $\alpha_{w,j}$, $\alpha_{l,j}$, $\alpha_{r,j}$: parameters	
	Fossil fuel–energy (FE) composite $C_j^{fe} = \min\ p_{ff,j} FF_j + p_{22} x_{22,j}$ s.t. $FE_j = \left(\alpha_{ff,j} FF_j^{\frac{\sigma_{fe}-1}{\sigma_{fe}}} + \alpha_{22,j} x_{22,j}^{\frac{\sigma_{fe}-1}{\sigma_{fe}}} \right)^{\frac{\sigma_{fe}}{\sigma_{fe}-1}}$	$FF_j = \frac{\alpha_{ff,j}^{\sigma_{fe}} p_{fe,j}^{\sigma_{fe}}}{p_{ff,j}^{\sigma_{fe}}} FE_j \qquad x_{22,j} = \frac{\alpha_{22,j}^{\sigma_{fe}} p_{fe,j}^{\sigma_{fe}}}{p_{22}^{\sigma_{FE}}} FE_j$ $C_j^{fe} = p_{fe,j} FE_j = \left[\left(\alpha_{ff,j}^{\sigma_{fe}} p_{ff,j}^{1-\sigma_{fe}} + \alpha_{22,j}^{\sigma_{fe}} p_{22}^{1-\sigma_{fe}} \right)^{\frac{1}{1-\sigma_{fe}}} \right] FE_j$
	C_j^{fe}: fossil fuel–energy cost, FF_j : input volume, $x_{22,j}$: electricity input volume, $p_{ff,j}$: unit cost of fossil fuel, p_{22}: price of electricity, $\alpha_{ff,j}$, $\alpha_{22,j}$: parameters, σ_{fe}: elasticity of substitution	
4	Fossil fuel (FF) composite $C_j^{ff} = \min(p_{17} + \tau^{CO2} \emptyset_{17}) x_{17,j} + p_{lq,j} LQ_j + p_{gs,j} GS_j$ s.t. $FF_j = \left(\alpha_{17,j} x_{17,j}^{\frac{\sigma_{ff}-1}{\sigma_{ff}}} + \alpha_{lq,j} LQ_j^{\frac{\sigma_{ff}-1}{\sigma_{ff}}} + \alpha_{gs,j} GS_j^{\frac{\sigma_{ff}-1}{\sigma_{ff}}} \right)^{\frac{\sigma_{ff}}{\sigma_{ff}-1}}$	$x_{17,j} = \frac{\alpha_{17,j}^{\sigma_{ff}} p_{ff,j}^{\sigma_{ff}}}{(p_{17} + \tau^{CO2} \emptyset_{17})^{\sigma_{ff}}} FF_j \qquad LQ_j = \frac{\alpha_{LQ,j}^{\sigma_{ff}} p_{ff,j}^{\sigma_{ff}}}{p_{lq,j}^{\sigma_{ff}}} FF_j$ $GS_j = \frac{\alpha_{GS,j}^{\sigma_{FF}} p_{FF,j}^{\sigma_{FF}}}{p_{GS,j}^{\sigma_{FF}}} FF_j \quad C_j^{ff} = p_{ff,j} FF_j = \left\{ \left[\alpha_{17,j}^{\sigma_{ff}} (p_{17} + \tau^{CO2} \emptyset_{17})^{1-\sigma_{ff}} + \alpha_{lq,j}^{\sigma_{ff}} p_{lq,j}^{1-\sigma_{ff}} + \alpha_{gs,j}^{\sigma_{ff}} p_{gs,j}^{1-\sigma_{ff}} \right]^{\frac{1}{1-\sigma_{ff}}} \right\} FF_j$
	C_j^{ff}: fossil fuel cost, $x_{17,j}$: coal input volume, LQ_j , GS_j : input volume, p_{17}: price of coal, τ^{CO2}: CO2 emission tax, $\emptyset_{17}$: emission factor, $p_{lq,j}, p_{gs,j}$: unit costs, $\alpha_{17,j}, \alpha_{lq,j}, \alpha_{gs,j}$: parameters, σ_{ff}: elasticity of substitution	
5	Liquid fossil fuel (LQ) composite $C_j^{lq} = \min\ (p_{18} + \tau^{CO2} \emptyset_{18}) x_{18,j} + (p_{19} + \tau^{CO2} \emptyset_{19}) x_{19,j}$ s.t. $LQ_j = \beta_{lq,j} x_{18,j}^{\alpha_{18,j}} x_{19,j}^{\alpha_{19,j}}$	$x_{18,j} = \frac{\alpha_{18,j} p_{lq,j}}{p_{18} + \tau^{CO2} \emptyset_{18}} LQ_j \quad x_{19,j} = \frac{\alpha_{19,j} p_{lq,j}}{p_{19} + \tau^{CO2} \emptyset_{19}} LQ_j$ $C_j^{lq} = p_{lq,j} LQ_j =$ $\left[\frac{1}{\beta_{lq,j}} \left(\frac{p_{18} + \tau^{CO2} \emptyset_{18}}{\alpha_{18,j}} \right)^{\alpha_{18,j}} \left(\frac{p_{19} + \tau^{CO2} \emptyset_{19}}{\alpha_{19,j}} \right)^{\alpha_{19,j}} \right] LQ_j$
	C_j^{lq}: liquid fossil fuel cost, $x_{18,j}, x_{19,j}$: oil and petroleum products input volume, p_{18}, p_{19}: price of oil and petroleum products, τ^{CO2}: CO2 emission tax, $\emptyset_{18}$,$\emptyset_{19}$: emission factor, $\beta_{lq,j}, \alpha_{18,j}, \alpha_{19,j}$: parameters	
	Gas fossil fuel (GS) composite $C_j^{gs} = \min\ (p_{20} + \tau^{CO2} \emptyset_{20}) x_{20,j} + (p_{21} + \tau^{CO2} \emptyset_{21}) x_{21,j}$ s.t. $GS_j = \beta_{gs,j} x_{20,j}^{\alpha_{20,j}} x_{21,j}^{\alpha_{21,j}}$	$x_{20,j} = \frac{\alpha_{20,j} p_{gs,j}}{p_{20} + \tau^{CO2} \emptyset_{20}} GS_j \quad x_{21,j} = \frac{\alpha_{21,j} p_{GS,j}}{p_{19} + \tau^{CO2} \emptyset_{19}} GS_j$ $C_j^{gs} = p_{gs,j} GS_j =$ $\left[\frac{1}{\beta_{gs,j}} \left(\frac{p_{20} + \tau^{CO2} \emptyset_{20}}{\alpha_{20,j}} \right)^{\alpha_{20,j}} \left(\frac{p_{21} + \tau^{CO2} \emptyset_{21}}{\alpha_{21,j}} \right)^{\alpha_{21,j}} \right] GS_j$
	C_j^{gs}: gas fossil fuel cost, $x_{20,j}, x_{21,j}$: gas and gas manufacturing input volume, p_{20}, p_{21}: price of gas and gas manufacturing, τ^{CO2}: CO2 emission tax, $\emptyset_{20}$,$\emptyset_{21}$: emission factor, $\beta_{gs,j}, \alpha_{20,j}, \alpha_{21,j}$: parameters	

Appendix B: Formulation of household's behaviors

Level	Cost minimization	Demand function
1	$\theta U = \min p_{fe,c}FE_c + p_{ne,c}NE_c$ s.t. $U = \min \left\{\frac{FE_c}{b_{fe}}, \frac{NE_c}{b_{ne}}\right\}$	$FE_c = b_{fe}U$ $NE_c = b_{ne}U$ $\theta U = (b_{fe}p_{fe,c} + b_{ne}p_{ne,c})U$
	U: utility level, b_{fe}, b_{ne}: share parameters, $p_{ff,c}, p_{ne,c}$: composite fossil fuel-energy goods and non-energy goods prices, respectively, K, W, L, R: endowments of capital, labor, land and natural resource, respectively, μ: full income, θ : unit expenditure or marginal utility of aggregate consumption = μ/U	
2	Fossil fuel–energy (FE_c) composite $(\mu_{fe})FE_c = \min p_{ff,c}FF_c + p_{22}x_{22,c}$ s.t. $FE_c = \left(\delta_{ff}FF_c^{\frac{\omega_{fe}-1}{\omega_{fe}}} + \delta_{22}x_{22,c}^{\frac{\omega_{fe}-1}{\omega_{fe}}}\right)^{\frac{\omega_{fe}}{\omega_{fe}-1}}$	$FF_c = \frac{\delta_{ff}^{\omega_{fe}}p_{fe,c}^{\omega_{fe}}}{p_{ff,c}^{\omega_{fe}}}FE_c$ $x_{22,c} = \frac{\delta_{22}^{\omega_{fe}}p_{fe,c}^{\omega_{fe}}}{p_{22}^{\omega_{FE}}}FE_c$ $p_{fe,c}FE_c = \left[\left(\delta_{ff}^{\omega_{fe}}p_{ff,c}^{1-\omega_{fe}} + \delta_{22}^{\omega_{fe}}p_{22}^{1-\omega_{fe}}\right)^{\frac{1}{1-\omega_{fe}}}\right]FE_c$
	FE_c : utility level from consuming fossil fuel-energy goods, δ_{ff}, δ_{22}: share parameters, $x_{22,c}$: consuming volume of electricity, $p_{ff,c}$: fossil fuel price, p_{22}: price of electricity, $p_{fe,c}$: unit expenditure of fossil fuel-energy goods, $\mu_{fe} := \mu - \mu_{ne}$	
	Non-energy (NE_c) goods composite $(\mu_{ne})NE_c = \sum_{i=1}^{16} p_i x_{i,c}$ s.t. $NE_c = \gamma_{ne}\prod_{i=1}^{16} x_{i,c}^{\delta_i}$	$x_{i,c} = \frac{\delta_i p_{ne,c}}{p_i}NE_c$ $p_{ne,c}NE_c = \left[\frac{1}{\gamma_{ne}}\prod_{i=1}^{16}\left(\frac{p_i}{\delta_{i,c}}\right)^{\delta_i}\right]NE_c$
	NE_c : utility level from consuming composite non-energy goods, γ_{ne}, δ_i : parameters, $x_{i,c}$: consuming volume of non-energy commodity i, p_i : price of commodity i, $\mu_{ne} := \mu - \mu_{fe}$	
3	Fossil fuel (FF_c) composite $(\mu_{ff})FF_c = (p_{17} + \tau^{CO2}\emptyset_{17})x_{17,c} + p_{lq,c}LQ_c + p_{gs,c}GS_c$ s.t. $FF_c = \left(\delta_{17}x_{17,c}^{\frac{\omega_{ff}-1}{\omega_{ff}}} + \delta_{lq}LQ_c^{\frac{\omega_{ff}-1}{\omega_{ff}}} + \delta_{gs}GS_c^{\frac{\omega_{ff}-1}{\omega_{ff}}}\right)^{\frac{\omega_{ff}}{\omega_{ff}-1}}$	$x_{17,c} = \frac{\delta_{17}^{\omega_{ff}}p_{ff,c}^{\omega_{ff}}}{(p_{17}+\tau^{CO2}\emptyset_{17})^{\omega_{ff}}}FF_c$ $LQ_c = \frac{\delta_{lq}^{\omega_{ff}}p_{ff,c}^{\omega_{ff}}}{p_{LQ,c}^{\omega_{FF}}}FF_c$ $GS_c = \frac{\delta_{gs}^{\omega_{ff}}p_{ff,c}^{\omega_{ff}}}{p_{gs,c}^{\omega_{ff}}}FF_c$ $p_{ff,c}FF_c = \left\{\left[\delta_{17}^{\omega_{ff}}(p_{17} + \tau^{CO2}\emptyset_{17})^{1-\omega_{ff}} + \delta_{lq}^{\omega_{ff}}p_{lq,c}^{1-\omega_{ff}} + \delta_{gs}^{\omega_{ff}}p_{gs,c}^{1-\omega_{ff}}\right]^{\frac{1}{1-\omega_{ff}}}\right\}FF_c$
	FF_c : utility from consuming composite fossil fuel, $\delta_{17}, \delta_{lq}, \delta_{gs}$: parameters, ω_{ff}: elasticity of substitution among fossil fuel, $x_{17,c}$: consuming volume of coal, LQ_c , GS_c : consuming volume of composite fossil fuel liquid and gas goods, respectively, p_{17}: price of coal, $p_{lq,c}, p_{gs,c}$: composite fossil fuel liquid and gas, respectively, $p_{ff,c}$: unit expenditure of composite fossil fuel goods, $\mu_{ff} := \mu_{fe} - p_{22}x_{22,c}$	
4	Liquid fossil fuel (LQ_c) composite $(\mu_{lq})LQ_c = (p_{18} + \tau^{CO2}\emptyset_{18})x_{18,c}^{\delta_{18}} + (p_{19} + \tau^{CO2}\emptyset_{19})x_{19,c}^{\delta_{19}}$ s.t. $LQ_c = \gamma_{lq}x_{18,c}^{\delta_{18}}x_{19,c}^{\delta_{19}}$	$x_{18,c} = \frac{\delta_{18}p_{lq,c}}{p_{18}+\tau^{CO2}\emptyset_{18}}LQ_c$ $x_{19,c} = \frac{\delta_{19}p_{lq,c}}{p_{19}+\tau^{CO2}\emptyset_{19}}LQ_c$ $p_{lq,c}LQ_c =$ $\left[\frac{1}{\gamma_{lq}}\left(\frac{p_{18} + \tau^{CO2}\emptyset_{18}}{\delta_{18}}\right)^{\delta_{18}}\left(\frac{p_{19} + \tau^{CO2}\emptyset_{19}}{\delta_{19}}\right)^{\delta_{19}}\right]LQ_c$
	LQ_c : utility from consuming composite fossil fuel liquid, $\gamma_{lq}, \delta_{18}, \delta_{19}$: parameters, $x_{18,c}, x_{19,c}$: consuming volume of oil and petroleum products, respectively, p_{18}, p_{19}: price of oil and petroleum products, respectively, $p_{lq,c}$: unit expenditure of composite fossil fuel liquid, $\mu_{lq} := \mu_{ff} - (p_{17} + \tau^{CO2}\emptyset_{17})x_{17,c} - \mu_{gs}$	
	Gas fossil fuel (GS_c) composite $(\mu_{gs})GS_c = (p_{20} + \tau^{CO2}\emptyset_{20})x_{20,c}^{\delta_{20}} + (p_{21} + \tau^{CO2}\emptyset_{21})x_{21,c}^{\delta_{21}}$ s.t. $GS_c = \gamma_{gs,c}x_{20,c}^{\delta_{20,c}}x_{21,c}^{\delta_{21,c}}$	$x_{20,c} = \frac{\delta_{20}p_{gs,c}}{p_{20}+\tau^{CO2}\emptyset_{20}}GS_c$ $x_{21,c} = \frac{\delta_{21}p_{gs,c}}{p_{21}+\tau^{CO2}\emptyset_{21}}GS_c$ $p_{gs,c}GS_c =$ $\left[\frac{1}{\gamma_{gs}}\left(\frac{p_{20} + \tau^{CO2}\emptyset_{20}}{\delta_{20}}\right)^{\delta_{20}}\left(\frac{p_{21} + \tau^{CO2}\emptyset_{21}}{\delta_{21}}\right)^{\delta_{21}}\right]GS_c$
	GS_c : utility from consuming fossil fuel gas, $\gamma_{gs}, \delta_{20}, \delta_{21}$: parameters, $x_{20,c}, x_{21,c}$: consuming volume of gas and gas manufacturing, p_{20}, p_{21}: price of oil and petroleum products, $p_{gs,c}$: unit expenditure of fossil fuel liquid, $\mu_{gs} := \mu_{ff} - (p_{17} + \tau^{CO2}\emptyset_{17})x_{17,c} - \mu_{lq}$	

Appendix C: The equilibrium conditions of the AIM/CGE Global Model

Zero profit conditions

1. Production of goods except energy:

$$py_{j,r} = \sum_{i=1}^{16} a_{i,j,r} p_{i,r} + a_{vae,j,r} \left(\alpha_{va,j,r}^{\sigma_{vae}} p_{va,j,r}^{1-\sigma_{vae}} + \alpha_{fe,j,r}^{\sigma_{vae}} p_{fe,j,r}^{1-\sigma_{vae}} \right)^{\frac{1}{1-\sigma_{vae}}} \tag{C1}$$

2. Armington aggregate of domestic and import goods:

$$p_{j,r}^{A} = \left[\beta_{i,r}^{\sigma_j^A} py_{i,r}^{1-\sigma_j^A} + \left(1 - \beta_{i,r}\right)^{\sigma_j^A} \left(\sum_{s} \beta_{i,s,r}^{\sigma_j^M} py_{i,s}^{1-\sigma_j^M} \right)^{\frac{1-\sigma_j^A}{1-\sigma_j^M}} \right]^{\frac{1}{1-\sigma_j^A}} \tag{C2}$$

3. Household consumption demand:

$$\theta_r = b_{fe,r} \left(\delta_{ff,r}^{\omega_{fe}} p_{ff,c,r}^{1-\omega_{fe}} + \delta_{22,r}^{\omega_{fe}} p_{22,r}^{1-\omega_{fe}} \right)^{\frac{1}{1-\omega_{fe}}} + \frac{b_{ne,r}}{\gamma_{ne,r}} \prod_{i=1}^{16} \left(\frac{p_{i,r}}{\delta_{i,r}} \right)^{\delta_{i,r}} \tag{C3}$$

Market clearance conditions

1. Capital:

$$K_r = \sum_{j=1}^{22} \alpha_{k,j,r}^{\sigma_{va}} \alpha_{va,j,r}^{\sigma_{vae}} a_{vae,j,r} \frac{p_{va,j}^{\sigma_{va}-\sigma_{vae}} p_{vae,j}^{\sigma_{vae}}}{p_{k,r}^{\sigma_{va}}} y_{j,r} \tag{C4}$$

2. Labor:

$$W_r = \sum_{j=1}^{22} \alpha_{w,j,r}^{\sigma_{va}} \alpha_{va,j,r}^{\sigma_{vae}} a_{vae,j,r} \frac{p_{va,j}^{\sigma_{va}-\sigma_{vae}} p_{vae,j}^{\sigma_{vae}}}{p_{w,r}^{\sigma_{va}}} y_{j,r} \tag{C5}$$

3. Land:

$$L_r = \sum_{j=1}^{22} \alpha_{l,j,r}^{\sigma_{va}} \alpha_{va,j,r}^{\sigma_{vae}} a_{vae,j,r} \frac{p_{va,j}^{\sigma_{va}-\sigma_{vae}} p_{vae,j}^{\sigma_{vae}}}{p_{l,r}^{\sigma_{va}}} y_{j,r} \tag{C6}$$

4. Resources:

$$R_r = \sum_{j=1}^{22} \alpha_{r,j,r}^{\sigma_{va}} \alpha_{va,j,r}^{\sigma_{vae}} a_{vae,j,r} \frac{p_{va,j}^{\sigma_{va}-\sigma_{vae}} p_{vae,j}^{\sigma_{vae}}}{p_{r,r}^{\sigma_{va}}} y_{j,r} \tag{C7}$$

5. Armington aggregate for non-energy goods (ne: sectors 1 to 16):

$$A_{ne,r} = \sum_{j=1}^{22} a_{ne,j,r} y_{j,r} + \frac{\delta_{ne,r} b_{ne,r}}{\gamma_{ne,r} p_{ne,r}} \prod_{i=1}^{16} \left(\frac{p_{i,r}}{\delta_{i,r}} \right)^{\delta_{i,r}} u_r \tag{C8}$$

6. Armington aggregate for coal (sector 17):

$$A_{17,r} = \sum_{j=1}^{22} \alpha_{17,j,r}^{\sigma_{ff}} \alpha_{ff,j,r}^{\sigma_{fe}} \alpha_{fe,j,r}^{\sigma_{vae}} a_{vae,j,r} \frac{p_{ff,j,r}^{\sigma_{ff}-\sigma_{fe}} p_{fe,j,r}^{\sigma_{fe}-\sigma_{vae}} p_{vae,j,r}^{\sigma_{vae}}}{(p_{17,r} + \tau^{CO2}\emptyset_{17})^{\sigma_{ff}}} y_{j,r} \tag{C9}$$

$$+ \delta_{17}^{\omega_{ff}} \delta_{ff}^{\omega_{fe}} b_{fe} \frac{p_{FF,c}^{\omega_{ff}-\omega_{fe}} p_{fe,c}^{\omega_{fe}}}{(p_{17} + \tau^{CO2}\emptyset_{17})^{\omega_{ff}}} u_r$$

7. Armington aggregate for liquid fossil fuel (lq: sectors 18 and 19):

$$A_{lq,r} = \sum_{j=1}^{22} \alpha_{lq,j} \alpha_{lq,j}^{\sigma_{ff}} \alpha_{ff,j}^{\sigma_{fe}} \alpha_{fe,j}^{\sigma_{vae}} a_{vae,j} \frac{p_{lq,j}^{1-\sigma_{ff}} p_{FF,j}^{\sigma_{ff}-\sigma_{fe}} p_{fe,j}^{\sigma_{fe}-\sigma_{vae}} p_{vae,j}^{\sigma_{vae}}}{p_{lq} + \tau^{CO2}\emptyset_{lq}} y_{j,r} \tag{C10}$$

$$+ \delta_{lq} \delta_{lq}^{\omega_{ff}} \delta_{ff}^{\omega_{fe}} b_{fe} \frac{p_{lq,c}^{1-\omega_{ff}} p_{ff,c}^{\omega_{ff}-\omega_{fe}} p_{fe,c}^{\omega_{fe}}}{p_{lq} + \tau^{CO2}\emptyset_{lq}} u_r$$

8. Armington aggregate for gas fossil fuel (gs: sector 20 and 21):

$$A_{gs,r} = \sum_{j=1}^{22} \alpha_{gs,j} \alpha_{gs,j}^{\sigma_{ff}} \alpha_{ff,j}^{\sigma_{fe}} \alpha_{fe,j}^{\sigma_{vae}} a_{vae,j} \frac{p_{gs,j}^{1-\sigma_{ff}} p_{ff,j}^{\sigma_{ff}-\sigma_{fe}} p_{fe,j}^{\sigma_{fe}-\sigma_{vae}} p_{vae,j}^{\sigma_{vae}}}{p_{gs} + \tau^{CO2}\emptyset_{gs}} y_{j,r} \tag{C11}$$

$$+ \delta_{gs} \delta_{gs}^{\omega_{ff}} \delta_{ff}^{\omega_{fe}} b_{fe} \frac{p_{gs,j}^{1-\omega_{ff}} p_{ff,j}^{\omega_{ff}-\omega_{fe}} p_{fe,j}^{\omega_{fe}}}{p_{gs} + \tau^{CO2}\emptyset_{gs}} u_r$$

9. Armington aggregate for electricity (sector 22):

$$A_{22,r} = \sum_{j=1}^{22} \alpha_{22,j}^{\sigma_{fe}} \alpha_{fe,j}^{\sigma_{vae}} a_{vae,j} \frac{p_{fe,j}^{\sigma_{fe}-\sigma_{vae}} p_{vae,j}^{\sigma_{vae}}}{p_{22}^{\sigma_{fe}}} y_{j,r} + \delta_{22}^{\omega_{fe}} b_{fe} \frac{p_{fe,c}^{\omega_{fe}}}{p_{22}^{\omega_{fe}}} u_r \tag{C12}$$

10. Household consumption:

$$u_r = \frac{\mu_r}{\theta_r} \tag{C13}$$

Income balance conditions

$$\mu_r = p_{k,r}K_r + p_{w,r}W_r + p_{l,r}L_r + p_{r,r}R_r + \tau^{CO2}Q_r^{CO2} \tag{C14}$$

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The Riemann Zeta-Function and Hecke Congruence Subgroups. II

Yoichi MOTOHASHI [1]

Abstract

This is a rework of our old file on an explicit spectral decomposition of the mean value

$$M_2(g;A) = \int_{-\infty}^{\infty} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|A\left(\tfrac{1}{2}+it\right)\right|^2 g(t)dt$$

that has been left unpublished since September 1994, though its summary account is given in [9] (see also [11, Section 4.6]); here

$$A(s) = \sum_n \alpha_n n^{-s}$$

is a finite Dirichlet series and g is assumed to be even, regular, real-valued on R, and of fast decay on a sufficiently wide horizontal strip. On this occasion we add greater details as well as a rigorous treatment of the Mellin transform

$$Z_2(s;A) = \int_1^{\infty} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 \left|A\left(\tfrac{1}{2}+it\right)\right|^2 t^{-s}dt$$

which was scantly touched on in [9]. In particular, we specify the location of its poles and respective residues, under a mild condition on the coefficients α_n .

Key Words : Zeta-function, Spectral theory, Hecke congruence groups

0. We shall proceed with an arbitrary A to a considerable extent but later restrict ourselves to the situation where α_n is supported by the set of square-free integers. This is solely to avoid certain technical complexities pertaining to Kloosterman sums associated with Hecke congruence subgroups which do not appear particularly worth dealing with thoroughly, for our present principal purpose is to look into the nature of $Z_2(s;A)$.

Our result on $Z_2(s;A)$ seems to allow us to have a glimpse of the nature of the plain sixth power moment

$$M_3(g;1) = \int_{-\infty}^{\infty} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^6 g(t)dt,$$

although we shall set out only certain ensuing problems which are to be solved before stating anything precisely. In fact, this motivation which was implicit in our original file was similar to that expressed in [4]. Our approach was, however, more explicit, being a natural extension of our treatment of the plain fourth moment $M_2(g;1)$ that was later published in [11].

As we noted at a few occasions, the reason of the success with $M_2(g;1)$ lies probably in the fact that the Eisenstein series in the framework of $\mathrm{SL}(2,\mathbb{R})$ is closely related to the product of two zeta-values and in that the group is of real rank one, with the observation that the later is reflected in that the integral for $M_2(g;1)$ is single (as is inferred from the arguments developed in e.g. [2][12]). Extrapolating this, we surmise that a proper formulation of the sixth moment of the zeta-function might be expressed instead in terms of a double integral, since the group $\mathrm{SL}(3,\mathbb{R})$ appears to be closely related to the product of three zeta-values and it is of real rank 2. Nevertheless, we shall consider $M_2(g;A)$, as it stands between the pure fourth and sixth moments and requires less machineries than the plausible direct approach to the sixth moment via the spectral theory of $L^2(\mathrm{PSL}(3,\mathbb{Z})\backslash\mathrm{PSL}(3,\mathbb{R}))$ such as proposed in [11, Section 5.4].

There are at least three ways for us to proceed along. The first is the argument that we took in [7][11], the second is a representation theoretic approach developed in [2], and the third is the one in [12] which is more representation theoretic and in fact generalizes to quite a wide extent. We shall take the first way, as we have

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indicated above, for it appears to be the most explicit and allows us to exploit best the peculiarity of our problem, i.e., the presence of the square of the zeta-function in place of the first power of an automorphic L-function. However, it should be stressed that the methods in [2] and [12] have a definite advantage over that in [7][11]; see REMARK 3 in Section 15 below.

Convention. *We shall assume throughout our discussion that there exist no exceptional eigenvalues for any Hecke congruence subgroup $\Gamma_0(q)$.*

Thus all spectral data κ_j should be understood to be real and non-negative. With this, we might not appear prudent enough, but actually our discussion of $Z_2(s;A)$ is not essentially affected by the assumption, though we are aware of the possible existence of poles in the interval $\left(\frac{1}{2},1\right)$.

REMARK 1. Readers are warned of a number of notational conflicts, none of which should, however, cause any serious misunderstanding. We remark also that our discussion contains details which must be often excessive for experts; nevertheless, we do this because our old file had been prepared for an abortive series of lectures to be given to beginners, and we want to keep the original style. By the way, there exists as well an abridged version of the file that was to be included in [11] as its sixth chapter, but the plan was put away because of a reason which we can no longer remember.

REMARK 2. We do not mention any of works on mean values of automorphic L-functions done in recent years, notably by D. Goldfeld and his colleagues, some of which in fact come close to our interest on $Z_2(s;A)$. This is solely due to our wish to keep ourselves within the framework of the unpublished file of ours; the necessary updating will be made in our relevant forthcoming works.

In passing, we stress that our work [8] (see also [11, Section 5.3]) on $Z_2(s;1)$ was done without any knowledge of the existence of A. Good's work [5] on the Mellin transform of the square of an arbitrary automorphic L-function. His argument depends on a clever choice of a Poincaré series, whereas ours exploits fully the peculiarity of the Riemann zeta-function as indicated above and produces results more explicit than his. We add that our reasoning extends beyond Good's situation. This is a consequence of our latest work [12] lying on the lines developed in [2], [7], and [11].

1. To begin with, we have

$$M_2(g;A)=\sum_{\substack{a,b,c\\(a,b)=1}}\frac{\alpha_{ac}\overline{\alpha_{bc}}}{c\sqrt{ab}}I_2(g;b/a),\tag{1.1}$$

where

$$I_2(g;b/a)=\int_{-\infty}^{\infty}\left|\zeta\left(\tfrac12+it\right)\right|^4(b/a)^{it}g(t)dt.\tag{1.2}$$

To study the latter we introduce

$$\begin{aligned}I(u,v,w,z;g;b/a)=-i\int_{(0)}&\zeta(u+t)\zeta(v-t)\\ \times\zeta(w+t)\zeta(z-t)&(b/a)^t g(-it)dt\end{aligned}\tag{1.3}$$

with $(a,b)=1$ and $\mathrm{Re}\,u,\ldots,\mathrm{Re}\,z>1$. Shifting the contour to (α) lying in the far right, we have

$$\begin{aligned}I(u,v,&w,z;g;b/a)=-i\int_{(\alpha)}\cdots dt\\ &+2\pi\Big\{\zeta(u+v-1)\zeta(v+w-1)\\ &\qquad\times\zeta(z-v+1)(b/a)^{v-1}g(i(1-v))\\ &\quad+\zeta(u+z-1)\zeta(w+z-1)\\ &\qquad\times\zeta(v-z+1)(b/a)^{z-1}g(i(1-z))\Big\}.\end{aligned}\tag{1.4}$$

Thus $I(u,v,w,z;g;b/a)$ is meromorphic throughout $\mathbb{C}^4$. With this, we assume that $\mathrm{Re}\,u,\ldots,\mathrm{Re}\,z<1$ and shift the last contour back to the original, getting

$$\begin{aligned}I(u,v,&w,z;g;b/a)=-i\int_{(0)}\cdots dt\\ &+2\pi\Big\{\zeta(u+v-1)\zeta(v+w-1)\\ &\qquad\times\zeta(z-v+1)(b/a)^{v-1}g(i(1-v))\\ &\quad+\zeta(u+z-1)\zeta(w+z-1)\\ &\qquad\times\zeta(v-z+1)(b/a)^{z-1}g(i(1-z))\\ &\quad+\zeta(w-u+1)\zeta(u+v-1)\\ &\qquad\times\zeta(u+z-1)(b/a)^{1-u}g(i(u-1))\\ &\quad+\zeta(u-w+1)\zeta(v+w-1)\\ &\qquad\times\zeta(w+z-1)(b/a)^{1-w}g(i(w-1))\Big\}.\end{aligned}\tag{1.5}$$

In the vicinity of the central point $p_{\frac12}=\left(\frac12,\frac12,\frac12,\frac12\right)$, the part in the braces is equal to

$$\begin{aligned}&\zeta(u+v-1)\zeta(v+w-1)\\ &\times\frac{1}{z-v}(1+c_E(z-v)+\cdots)(b/a)^{v-1}g(i(1-v))\\ &+\zeta(u+v-1)\left(1+\frac{\zeta'}{\zeta}(u+v-1)(z-v)+\cdots\right)\end{aligned}$$

$$
\begin{aligned}
&\times \zeta(v+w-1)\left(1+\frac{\zeta'}{\zeta}(v+w-1)(z-v)+\cdots\right)\\
&\times \frac{1}{v-z}(1+c_E(v-z)+\cdots)\\
&\times (b/a)^{v-1}\left(1+(\log b/a)(z-v)+\cdots\right)\\
&\times g(i(1-v))\left(1-i\frac{g'}{g}(i(1-v))(z-v)+\cdots\right)\\
&+\frac{1}{w-u}(1+c_E(w-u)+\cdots)\zeta(u+v-1)\\
&\times \zeta(u+z-1)(b/a)^{1-u}g(i(u-1))\\
&+\frac{1}{u-w}(1+c_E(u-w)+\cdots)\zeta(u+v-1)\\
&\times\left(1+\frac{\zeta'}{\zeta}(u+v-1)(w-u)+\cdots\right)\\
&\times \zeta(u+z-1)\left(1+\frac{\zeta'}{\zeta}(u+z-1)(v-u)+\cdots\right)\\
&\times (b/a)^{1-u}\left(1+(\log b/a)(u-w)+\cdots\right)\\
&\times g(i(u-1))\left(1+i\frac{g'}{g}(i(u-1))(w-u)+\cdots\right)\\
=&\,\zeta(u+v-1)\zeta(v+w-1)(b/a)^{v-1}g(i(1-v))\\
&\times\left(2c_E-\frac{\zeta'}{\zeta}(u+v-1)-\frac{\zeta'}{\zeta}(v+w-1)\right.\\
&\left.-\log b/a+\frac{g'}{g}(i(1-v))\right)+O(|z-v|)\\
&+\zeta(u+v-1)\zeta(u+z-1)(b/a)^{1-u}g(i(u-1))\\
&\times\left(2c_E-\frac{\zeta'}{\zeta}(u+v-1)-\frac{\zeta'}{\zeta}(u+z-1)\right.\\
&\left.+\log b/a+\frac{g'}{g}(i(u-1))\right)+O(|u-w|),
\end{aligned}\tag{1.6}
$$

where c_E is the Euler constant. Hence, in particular, $I(u,v,w,z;g;b/a)$ is regular in a neighborhood of $p_{\frac{1}{2}}$, and we get

$$
\begin{aligned}
I_2(g;b/a)=&\,I\left(p_{\frac{1}{2}};g;b/a\right)\\
&-\frac{\pi}{2}(b/a)^{-1/2}g\left(\tfrac{1}{2}i\right)\\
&\times\left\{2c_E-2\log 2\pi-\log(b/a)+i\frac{g'}{g}\left(\tfrac{1}{2}i\right)\right\}\\
&-\frac{\pi}{2}(a/b)^{-1/2}g\left(\tfrac{1}{2}i\right)\\
&\times\left\{2c_E-2\log 2\pi-\log(a/b)-i\frac{g'}{g}\left(\tfrac{1}{2}i\right)\right\}.
\end{aligned}\tag{1.7}
$$

The last two terms can be regarded as practically negligible.

2. On the other hand, we have, in the region of absolute convergence,

$$
\begin{aligned}
&I(u,v,w,z;g;b/a)\\
&=\sum_{k,l,m,n}\frac{1}{k^u l^v m^w n^z}\hat{g}\left(\log\frac{bln}{akm}\right)\\
&=\zeta(u+v)\sum_{\substack{k,l\\(k,l)=1}}\frac{1}{k^u l^v}\sum_{m,n}\frac{1}{m^w n^z}\hat{g}\left(\log\frac{bln}{akm}\right),
\end{aligned}\tag{2.1}
$$

where $\hat{g}$ is the Fourier transform of g; and $(ak,bl)=(a,l)\cdot(b,k)=c\cdot d$, say; note that $(a,b)=1$. We have

$$
\begin{aligned}
&I(u,v,w,z;g;b/a)\\
&=\zeta(u+v)\sum_{c|a,d|b}\frac{1}{c^v d^u}\sum_{\substack{k,l,(k,l)=1\\(a/c,l)=1,(b/d,k)=1}}\frac{1}{k^u l^v}\\
&\times\sum_{m,n}\frac{1}{m^w n^z}\hat{g}\left(\log\frac{bln/d}{akm/c}\right)\\
&=\zeta(u+v)\frac{1}{a^v b^u}\sum_{c|a,d|b}c^v d^u\sum_{\substack{k,l,(k,l)=1\\(c,l)=1,(d,k)=1}}\frac{1}{k^u l^v}\\
&\times\sum_{m,n}\frac{1}{m^w n^z}\hat{g}\left(\log\frac{dln}{ckm}\right)\\
&=\zeta(u+v)\frac{1}{a^v b^u}\sum_{c|a,d|b}c^v d^u\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^u l^v}\\
&\times\sum_{m,n}\frac{1}{m^w n^z}\hat{g}\left(\log\frac{dln}{ckm}\right)\\
&=\zeta(u+v)\frac{1}{a^v b^u}\sum_{c|a,d|b}c^v d^u J(u,v,w,z;g;d/c),
\end{aligned}\tag{2.2}
$$

say.

Then we apply the dissection:

$$
\begin{aligned}
&J(u,v,w,z;g;d/c)\\
&=\{J_0+J_++J_-\}(u,v,w,z;g;d/c),
\end{aligned}\tag{2.3}
$$

where $J_-(u,v,w,z;g;d/c)=J_+(v,u,z,w;g;c/d)$ and

$$
\begin{aligned}
&J_0(u,v,w,z;g;d/c)\\
&=\hat{g}(0)\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^u l^v}\sum_{\substack{m,n\\ckm=dln}}\frac{1}{m^w n^z},
\end{aligned}\tag{2.4}
$$

$$
\begin{aligned}
&J_+(u,v,w,z;g;d/c)\\
&=\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^u l^v}\sum_{\substack{m,n\\ckm>dln}}\frac{1}{m^w n^z}\hat{g}\left(\log\frac{dln}{ckm}\right).
\end{aligned}\tag{2.5}
$$

We have

$$
\begin{aligned}
&J_0(u,v,w,z;g;d/c)\\
&=\hat{g}(0)\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^u l^v}\sum_n\frac{1}{(dln)^w(ckn)^z}\\
&=\hat{g}(0)c^{-z}d^{-w}\zeta(w+z)\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^{u+z}l^{v+w}}
\end{aligned}
$$

$$= \hat g(0)c^{-z}d^{-w}\zeta(w+z)\sum_{k,l}\frac{1}{k^{u+z}l^{v+w}}\sum_{r|(ck,dl)}\mu(r)$$
$$= \hat g(0)c^{-z}d^{-w}\zeta(w+z)\sum_{r}\mu(r)$$
$$\times\sum_{r/(c,r)|k}\frac{1}{k^{u+z}}\sum_{r/(d,r)|l}\frac{1}{l^{v+w}}$$
$$= \hat g(0)c^{-z}d^{-w}\zeta(w+z)\zeta(u+z)\zeta(v+w)$$
$$\times\sum_{r}\mu(r)((c,r)/r)^{u+z}((d,r)/r)^{v+w}$$
$$= \hat g(0)c^{-z}d^{-w}\zeta(w+z)\zeta(u+z)\zeta(v+w)$$
$$\times\prod_{p\nmid cd}\left(1-\frac{1}{p^{u+v+w+z}}\right)\prod_{p|c}\left(1-\frac{1}{p^{v+w}}\right)$$
$$\times\prod_{p|d}\left(1-\frac{1}{p^{u+z}}\right), \tag{2.6}$$

where p denotes a generic prime and the condition $(c,d)=1$ has been used. The contribution of $J_0(u,v,w,z;g;d/c)$ to $I(u,v,w,z;g;b/a)$ is thus equal to

$$\hat g(0)a^{-v}b^{-u}\frac{\zeta(u+v)\zeta(u+z)\zeta(w+v)\zeta(w+z)}{\zeta(u+v+w+z)}$$
$$\times\left\{\sum_{c|a}c^{v-z}\prod_{p|c}\frac{1-\dfrac{1}{p^{v+w}}}{1-\dfrac{1}{p^{u+v+w+z}}}\right\}$$
$$\times\left\{\sum_{d|b}d^{u-w}\prod_{p|d}\frac{1-\dfrac{1}{p^{u+z}}}{1-\dfrac{1}{p^{u+v+w+z}}}\right\}. \tag{2.7}$$

3. Next, we shall consider the non-diagonal part J_+. We have

$$J_+(u,v,w,z;g;d/c) = \sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^u l^v}$$
$$\times\sum_f\sum_{\substack{m,n\\ckm=dln+f}}\frac{1}{m^w n^z}\hat g\left(\log\frac{dln}{ckm}\right)$$
$$=\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^u l^v}$$
$$\times\sum_f\sum_{\substack{m,n\\ckm=dln+f}}\frac{(ck)^w}{(ckm)^w n^z}\hat g\left(\log\frac{dln}{ckm}\right)$$
$$=\sum_{\substack{k,l\\(ck,dl)=1}}\frac{(ck)^w}{k^u l^v}$$
$$\times\sum_f\sum_{\substack{n\\dln+f\equiv 0 \bmod ck}}\frac{1}{(dln+f)^w n^z}$$
$$\times\hat g\left(\log\frac{dln}{dln+f}\right)$$
$$=(c/d)^w\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^{u-w}l^{v+w}}$$
$$\times\sum_f\sum_{n\equiv-\overline{dl}f \bmod ck}\frac{1}{n^{w+z}}\left(1+\frac{f}{dln}\right)^{-w}$$
$$\times\hat g\left(\log\left(1+\frac{f}{dln}\right)\right). \tag{3.1}$$

We then introduce the Mellin transform

$$g^*(s,w)=\int_0^\infty\hat g(\log(1+x))\frac{x^{s-1}}{(1+x)^w}dx$$
$$=\Gamma(s)\int_{-\infty}^{\infty}\frac{\Gamma(w-s+it)}{\Gamma(w+it)}g(t)dt, \tag{3.2}$$

provided $\mathrm{Re}\,w>\mathrm{Re}\,s>0$. Shifting the last contour downward appropriately, we see that $g^*(s,w)/\Gamma(s)$ is entire in s,w; and an upward shift gives that $g^*(s,w)$ is of rapid decay in s as far as w and $\mathrm{Re}\,s$ are bounded (see [11, Lemma 4.1]). In particular, we have

$$J_+(u,v,w,z;g;d/c)$$
$$=\frac{(c/d)^w}{2\pi i}\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^{u-w}l^{v+w}}$$
$$\times\sum_f\sum_{n\equiv-\overline{dl}f \bmod ck}\frac{1}{n^{w+z}}$$
$$\times\int_{(\eta)}g^*(s,w)\left(\frac{f}{dln}\right)^{-s}ds, \tag{3.3}$$

with $\eta>0$, which converges absolutely if

$$\eta>1,\ \mathrm{Re}\,u>\mathrm{Re}\,w+1,$$
$$\mathrm{Re}\,(v+w)>\eta+1,\ \mathrm{Re}\,(w+z)>\eta+1. \tag{3.4}$$

On this condition, we have

$$J_+(u,v,w,z;g;d/c)$$
$$=\frac{(c/d)^w}{2\pi i}\int_{(\eta)}g^*(s,w)d^s\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^{u-w}l^{v+w-s}}$$
$$\times\sum_f\frac{1}{f^s}\sum_{n\equiv-\overline{dl}f \bmod ck}\frac{1}{n^{w+z-s}}\,ds$$
$$=\frac{c^{-z}d^{-w}}{2\pi i}\int_{(\eta)}g^*(s,w)(cd)^s\sum_{\substack{k,l\\(ck,dl)=1}}\frac{1}{k^{u+z-s}l^{v+w-s}}$$
$$\times\sum_f\frac{1}{f^s}\zeta\left(w+z-s,-\frac{\overline{dl}f}{ck}\right)ds, \tag{3.5}$$

where $\zeta(s,\omega)$ is the Hurwitz zeta-function. Classifying l into residue classes mod ck, we have

$$\begin{aligned}
&J_+(u,v,w,z;g;d/c)\\
&=\frac{c^{-z}d^{-w}}{2\pi i}\int_{(\eta)} g^*(s,w)(cd)^s \sum_{(k,d)=1}\frac{1}{k^{u+z-s}}\\
&\times\sum_f \frac{1}{f^s}\sum_{\substack{h=1\\(h,ck)=1}}^{ck}\ \sum_{l\equiv h \bmod ck}\frac{1}{l^{v+w-s}}\\
&\times\zeta\left(w+z-s,-\frac{\overline{dh}f}{ck}\right)ds\\
&=\frac{c^{-v-w-z}d^{-w}}{2\pi i}\int_{(\eta)} g^*(s,w)(c^2d)^s\\
&\times\sum_{(k,d)=1}\frac{1}{k^{u+v+w+z-2s}}\\
&\times\sum_f\frac{1}{f^s}\sum_{\substack{h=1\\(h,ck)=1}}^{ck}\zeta\left(v+w-s,\frac{h}{ck}\right)\\
&\times\zeta\left(w+z-s,-\frac{\overline{dh}f}{ck}\right)ds. \qquad (3.6)
\end{aligned}$$

4. We are going to shift the last contour. To this end we assume that there exists a large η_1 such that $\eta_1>\eta+1$ and

$$\begin{aligned}
&\mathrm{Re}\,(v+w)<\eta_1,\ \mathrm{Re}\,(w+z)<\eta_1,\\
&\mathrm{Re}\,(u+v+w+z)>2(\eta_1+1).
\end{aligned}\qquad (4.1)$$

On this and $1<\mathrm{Re}\,s<\eta_1+\varepsilon$ with a small $\varepsilon>0$, the sum

$$\begin{aligned}
&\sum_{(k,d)=1}\frac{1}{k^{u+v+w+z-2s}}\\
&\times\sum_f\frac{1}{f^s}\sum_{\substack{h=1\\(h,ck)=1}}^{ck}\zeta\left(v+w-s,\frac{h}{ck}\right)\\
&\times\zeta\left(w+z-s,-\frac{\overline{dh}f}{ck}\right) \qquad (4.2)
\end{aligned}$$

is a meromorphic function of the five complex variables. To see this we note that for any finite s

$$\zeta(s,\omega)\ll|s-1|^{-1}+\omega^{-\mathrm{Re}\,s}\qquad(0<\omega\le1),\quad (4.3)$$

as it follows via an application of partial summation to the Dirichlet series defining $\zeta(s,\omega)$. Thus (4.2) is, provided neither $v+w-s$ nor $w+z-s$ is too close to 1,

$$\begin{aligned}
&\ll\sum_k k^{\mathrm{Re}\,(2s-u-v-w-z)+1}\\
&\quad\times\left(1+k^{\mathrm{Re}\,(v+w-s)}\right)\left(1+k^{\mathrm{Re}\,(w+z-s)}\right)\\
&=\sum_k\Big\{k^{\mathrm{Re}\,(2s-u-v-w-z)+1}+k^{\mathrm{Re}\,(s-u-z)+1}\\
&\quad+k^{\mathrm{Re}\,(s-u-v)+1}+k^{\mathrm{Re}\,(w-u)+1}\Big\}, \qquad (4.4)
\end{aligned}$$

in which we have

$$\begin{aligned}
&\mathrm{Re}\,(2s-u-v-w-z)+1\\
&\ <\mathrm{Re}\,(2s)-2(\eta_1+1)+1,\\
&\mathrm{Re}\,(s-u-z)+1\\
&\ =\mathrm{Re}\,(s)-\mathrm{Re}\,(u+v+w+z)+\mathrm{Re}\,(v+w)+1\\
&\ <\mathrm{Re}\,(s)-2(\eta_1+1)+\eta_1+1,\\
&\mathrm{Re}\,(s-u-v)+1\\
&\ =\mathrm{Re}\,(s)-\mathrm{Re}\,(u+v+w+z)+\mathrm{Re}\,(w+z)+1\\
&\ <\mathrm{Re}\,(s)-2(\eta_1+1)+\eta_1+1,\\
&\mathrm{Re}\,(w-u)+1\\
&\ =\mathrm{Re}\,(v+w)+\mathrm{Re}\,(w+z)-\mathrm{Re}\,(u+v+w+z)+1\\
&\ <\eta_1+\eta_1-2(\eta_1+1)+1; \qquad (4.5)
\end{aligned}$$

and the assertion follows.

With this, we shift the contour in (3.6) to (η_1). We encounter poles at $s=w+z-1, v+w-1$; we may assume without loss of generality that they do not coincide. Before computing the residues, we note that

$$\begin{aligned}
&\sum_{\substack{h=1\\(h,q)=1}}\zeta(s,hm/q)\\
&=\zeta(s)\sum_{\delta|q}\delta\mu(q/\delta)(\delta/(\delta,m))^{s-1}. \qquad (4.6)
\end{aligned}$$

To show this we use the functional equation

$$\begin{aligned}
&\zeta(s,\omega)=2(2\pi)^{s-1}\Gamma(1-s)\\
&\quad\times\sum_n\sin\left(\tfrac12\pi s+2\pi n\omega\right)n^{s-1}\quad(\mathrm{Re}\,s<0).\quad(4.7)
\end{aligned}$$

Thus, for $\mathrm{Re}\,s<0$,

$$\begin{aligned}
&\sum_{\substack{h=1\\(h,q)=1}}^{q}\zeta(s,hm/q)=2\Gamma(1-s)(2\pi)^{s-1}\\
&\quad\times\sum_n\sum_{\substack{h=1\\(h,q)=1}}^{q}\sin\left(\tfrac12\pi s+2\pi\frac{h}{q}mn\right)n^{s-1}\\
&=2\Gamma(1-s)(2\pi)^{s-1}\sin\left(\tfrac12\pi s\right)\sum_n c_q(mn)n^{s-1}\\
&=2\Gamma(1-s)(2\pi)^{s-1}\sin\left(\tfrac12\pi s\right)\\
&\quad\times\sum_n n^{s-1}\sum_{\delta|(q,mn)}\delta\mu(q/\delta)\\
&=2\Gamma(1-s)(2\pi)^{s-1}\sin\left(\tfrac12\pi s\right)\zeta(1-s)\\
&\quad\times\sum_{\delta|q}\delta\mu(q/\delta)(\delta/(\delta,m))^{s-1}, \qquad (4.8)
\end{aligned}$$

with the Ramanujan sum c_q mod q; and (4.6) follows via the functional equation for ζ.

Let us compute the residue at $s = w + z - 1$. This is equal to

$$
\begin{aligned}
&2\pi i(c^2d)^{w+z-1}g^*(w+z-1,w)\\
&\quad\times\sum_{(k,d)=1}\frac{1}{k^{u+v-w-z+2}}\\
&\quad\times\sum_f\frac{1}{f^{w+z-1}}\sum_{\substack{h=1\\(h,ck)=1}}^{ck}\zeta\left(v-z+1,\frac{h}{ck}\right)\\
=&2\pi i(c^2d)^{w+z-1}g^*(w+z-1,w)\\
&\quad\times\zeta(w+z-1)\zeta(v-z+1)\\
&\quad\times\sum_{(k,d)=1}\frac{1}{k^{u+v-w-z+2}}\sum_{\delta|ck}\delta^{v-z+1}\mu(ck/\delta)\\
=&2\pi i c^{2w+v+z-1}d^{w+z-1}g^*(w+z-1,w)\\
&\quad\times\zeta(w+z-1)\zeta(v-z+1)\\
&\quad\times\sum_{(k,d)=1}\frac{1}{k^{u-w+1}}\prod_{p|ck}\left(1-\frac{1}{p^{v-z+1}}\right)\\
=&2\pi i c^{2w+v+z-1}d^{w+z-1}g^*(w+z-1,w)\\
&\quad\times\zeta(w+z-1)\zeta(v-z+1)\\
&\quad\times\prod_{p\nmid cd}\left(1+\frac{1}{p^{u-w+1}}\left(1-\frac{1}{p^{v-z+1}}\right)\left(1-\frac{1}{p^{u-w+1}}\right)^{-1}\right)\\
&\quad\times\prod_{p|c}\left(1-\frac{1}{p^{v-z+1}}\right)\left(1-\frac{1}{p^{u-w+1}}\right)^{-1}.
\end{aligned}
\tag{4.9}
$$

Returning to (3.6), we see that the contribution of the residue to $J_+(u,v,w,z,;g;d/c)$ is

$$
\begin{aligned}
&c^{w-1}d^{z-1}g^*(w+z-1,w)\\
&\quad\times\frac{\zeta(v-z+1)\zeta(w+z-1)\zeta(u-w+1)}{\zeta(u+v-w-z+2)}\\
&\quad\times\prod_{p|c}\left(\frac{1-\dfrac{1}{p^{v-z+1}}}{1-\dfrac{1}{p^{u+v-w-z+2}}}\right)\\
&\quad\times\prod_{p|d}\left(\frac{1-\dfrac{1}{p^{u-w+1}}}{1-\dfrac{1}{p^{u+v-w-z+2}}}\right).
\end{aligned}
\tag{4.10}
$$

5. The residue at $s = v + w - 1$ is equal to

$$
\begin{aligned}
&2\pi i(c^2d)^{v+w-1}g^*(v+w-1,w)\\
&\quad\times\sum_{(k,d)=1}\frac{1}{k^{u-v-w+z+2}}\\
&\quad\times\sum_f\frac{1}{f^{v+w-1}}\sum_{\substack{h=1\\(h,ck)=1}}^{ck}\zeta\left(z-v+1,-\frac{\overline{dh}f}{ck}\right)\\
=&2\pi i(c^2d)^{v+w-1}g^*(v+w-1,w)\\
&\quad\times\zeta(z-v+1)\sum_{(k,d)=1}\frac{1}{k^{u-v-w+z+2}}\\
&\quad\times\sum_{\delta|ck}\delta^{z-v+1}\mu(ck/\delta)\sum_f\frac{(f,\delta)^{v-z}}{f^{v+w-1}},
\end{aligned}
\tag{5.1}
$$

as before. Here

$$
\begin{aligned}
&\sum_f\frac{(f,\delta)^{v-z}}{f^{v+w-1}}\\
=&\sum_f\frac{1}{f^{v+w-1}}\sum_{\lambda|(f,\delta)}\lambda^{v-z}\prod_{p|\lambda}\left(1-\frac{1}{p^{v-z}}\right)\\
=&\zeta(v+w-1)\sum_{\lambda|\delta}\frac{1}{\lambda^{w+z-1}}\prod_{p|\lambda}\left(1-\frac{1}{p^{v-z}}\right),
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
&\sum_{\delta|ck}\delta^{z-v+1}\mu(ck/\delta)\sum_f\frac{(f,\delta)^{v-z}}{f^{v+w-1}}\\
=&\zeta(v+w-1)\sum_{\lambda|ck}\frac{1}{\lambda^{w+z-1}}\prod_{p|\lambda}\left(1-\frac{1}{p^{v-z}}\right)\\
&\quad\times\sum_{\delta|(ck)/\lambda}(\delta\lambda)^{z-v+1}\mu(ck/\delta\lambda)\\
=&(ck)^{z-v+1}\zeta(v+w-1)\sum_{\lambda|ck}\frac{1}{\lambda^{w+z-1}}\\
&\quad\times\prod_{p|\lambda}\left(1-\frac{1}{p^{v-z}}\right)\prod_{p|(ck)/\lambda}\left(1-\frac{1}{p^{z-v+1}}\right).
\end{aligned}
\tag{5.3}
$$

Thus

$$
\begin{aligned}
&\sum_{(k,d)=1}\frac{1}{k^{u-v-w+z+2}}\sum_{\delta|ck}\delta^{z-v+1}\mu(ck/\delta)\sum_f\frac{(f,\delta)^{v-z}}{f^{v+w-1}}\\
&=c^{z-v+1}\zeta(v+w-1)\sum_{(k,d)=1}\frac{1}{k^{u-w+1}}\sum_{\lambda|ck}\frac{1}{\lambda^{z+w-1}}\\
&\quad\times\prod_{p|\lambda}\left(1-\frac{1}{p^{v-z}}\right)\prod_{p|(ck)/\lambda}\left(1-\frac{1}{p^{z-v+1}}\right)\\
&=c^{z-v+1}\zeta(v+w-1)\sum_{(\lambda,d)=1}\frac{(c,\lambda)^{u-w+1}}{\lambda^{u+z}}\\
&\quad\times\prod_{p|\lambda}\left(1-\frac{1}{p^{v-z}}\right)\sum_{(k,d)=1}\frac{1}{k^{u-w+1}}\\
&\quad\times\prod_{p|(ck)/(c,\lambda)}\left(1-\frac{1}{p^{z-v+1}}\right)\\
&=c^{z-v+1}\zeta(v+w-1)\\
&\quad\times\sum_{(\lambda,d)=1}\frac{(c,\lambda)^{u-w+1}}{\lambda^{u+z}}\prod_{p|\lambda}\left(1-\frac{1}{p^{v-z}}\right)
\end{aligned}
$$

$$\times \prod_{p\nmid(cd)/(c,\lambda)} \left(1+\frac{1}{p^{u-w+1}}\left(1-\frac{1}{p^{z-v+1}}\right)\right.$$
$$\left.\times\left(1-\frac{1}{p^{u-w+1}}\right)^{-1}\right)$$
$$\times \prod_{p|c/(c,\lambda)} \left(1-\frac{1}{p^{z-v+1}}\right)\left(1-\frac{1}{p^{u-w+1}}\right)^{-1}$$
$$= c^{z-v+1}\frac{\zeta(v+w-1)\zeta(u-w+1)}{\zeta(u-v-w+z+2)}$$
$$\times \prod_{p|d} \left(\frac{1-\dfrac{1}{p^{u-w+1}}}{1-\dfrac{1}{p^{u-v-w+z+2}}}\right)$$
$$\times \sum_{(\lambda,d)=1} \frac{(c,\lambda)^{u-w+1}}{\lambda^{u+z}} \prod_{p|\lambda}\left(1-\frac{1}{p^{v-z}}\right)$$
$$\times \prod_{p|c/(c,\lambda)} \left(\frac{1-\dfrac{1}{p^{z-v+1}}}{1-\dfrac{1}{p^{u-v-w+z+2}}}\right). \tag{5.4}$$

In the last sum we write $\lambda=\lambda_1\lambda_2$ with $(\lambda_1,c)=1$ and $\lambda_2|c^\infty$; and we see that the sum is equal to

$$\frac{\zeta(u+z)}{\zeta(u+v)}\prod_{p|cd}\left(\frac{1-\dfrac{1}{p^{u+z}}}{1-\dfrac{1}{p^{u+v}}}\right)$$
$$\times \prod_{p^\beta\|c}\left(\sum_{j=0}^\infty \frac{p^{(u-w+1)\min(\beta,j)}}{p^{j(u+z)}}\left(1-\frac{1}{p^{v-z}}\right)^{\xi(j)}\right.$$
$$\left.\times\left(\frac{1-\dfrac{1}{p^{z-v+1}}}{1-\dfrac{1}{p^{u-v-w+z+2}}}\right)^{\xi(\beta-\min(\beta,j))}\right), \tag{5.5}$$

where $1-\xi$ is the unit measure placed at the origin. One could compute the last sum into a finite expression.

The contribution of the residue at $s=v+w-1$ to $J_+(u,v,w,z;g;d/c)$ is equal to

$$c^{w-1}d^{v-1}g^*(v+w-1,w)$$
$$\times \frac{\zeta(z-v+1)\zeta(v+w-1)\zeta(u-w+1)\zeta(u+z)}{\zeta(u+v)\zeta(u-v-w+z+2)}$$
$$\times\prod_{p|cd}\left(\frac{1-\dfrac{1}{p^{u+z}}}{1-\dfrac{1}{p^{u+v}}}\right)\prod_{p|d}\left(\frac{1-\dfrac{1}{p^{u-w+1}}}{1-\dfrac{1}{p^{u-v-w+z+2}}}\right)$$
$$\times\prod_{p^\beta\|c}(\cdots), \tag{5.6}$$

where the last product is as in (5.5).

6. Now let us turn to

$$J_+^*(u,v,w,z;g;d/c)$$
$$=\frac{c^{-v-w-z}d^{-w}}{2\pi i}\int_{(\eta_1)} g^*(s,w)(c^2d)^s$$
$$\times\sum_{(k,d)=1}\frac{1}{k^{u+v+w+z-2s}}\sum_f\frac{1}{f^s}$$
$$\times\sum_{\substack{h=1\\(h,ck)=1}}^{ck}\zeta\left(v+w-s,\frac{h}{ck}\right)$$
$$\times\zeta\left(w+z-s,-\frac{\overline{dh}f}{ck}\right)ds, \tag{6.1}$$

where (4.1) holds. On noting that $\mathrm{Re}\,(v+w-s)<0$, $\mathrm{Re}\,(w+z-s)<0$, we appeal to the functional equation (4.7). Then the last double sum is equal to

$$4\frac{\Gamma(1+s-v-w)\Gamma(1+s-w-z)}{(2\pi)^{2+2s-v-2w-z}}$$
$$\times\sum_{f,m,n} m^{v+w-s-1}n^{w+z-1}(fn)^{-s}$$
$$\times\sum_{\substack{h=1\\(h,ck)=1}}^{ck}\sin\left(\tfrac12\pi(v+w-s)+2\pi\frac{m}{ck}h\right)$$
$$\times\sin\left(\tfrac12\pi(w+z-s)-2\pi\frac{fn}{ck}\overline{dh}\right)$$
$$=2\frac{\Gamma(1+s-v-w)\Gamma(1+s-w-z)}{(2\pi)^{2+2s-v-2w-z}}$$
$$\times\sum_{f,m,n} m^{v+w-s-1}n^{w+z-1}(fn)^{-s}$$
$$\times\Big\{\cos\left(\tfrac12\pi(v-z)\right)S\big(m,\overline{d}fn;ck\big)$$
$$-\cos\left(\tfrac12\pi(v+2w+z-2s)\right)S\big(m,-\overline{d}fn;ck\big)\Big\}$$
$$=2\frac{\Gamma(1+s-v-w)\Gamma(1+s-w-z)}{(2\pi)^{2+2s-v-2w-z}}$$
$$\times\sum_{m,n} m^{v+w-s-1}n^{-s}\sigma_{w+z-1}(n)$$
$$\times\Big\{\cos\left(\tfrac12\pi(v-z)\right)S\big(m,\overline{d}n;ck\big)$$
$$-\cos\left(\tfrac12\pi(v+2w+z-2s)\right)S\big(m,-\overline{d}n;ck\big)\Big\}, \tag{6.2}$$

where S is the ordinary Kloosterman sum, and $\sigma_\tau(n)=\sum_{\lambda|n}\lambda^\tau$.

Thus

$$J_+^*(u,v,w,z;g;d/c)$$
$$=\frac{c^u d^{\frac12(u+v-w+z)}}{\pi i(2\pi)^{u-w+1}}$$
$$\times\sum_{m,n} m^{\frac12(v+w-u-z-1)}n^{-\frac12(u+v+w+z-1)}\sigma_{w+z-1}(n)$$

$$\times \sum_{(k,d)=1} \frac{1}{ck\sqrt{d}} \int_{(\eta_1)} \Big\{ \cos\left(\tfrac12\pi(v-z)\right) S(m,\bar{d}n;ck)$$
$$- \cos\left(\tfrac12\pi(v+2w+z-2s)\right) S(m,-\bar{d}n;ck) \Big\}$$
$$\times \Gamma(1+s-v-w)\Gamma(1+s-w-z)$$
$$\times g^*(s,w) \left(\frac{2\pi\sqrt{mn}}{ck\sqrt{d}}\right)^{u+v+w+z-2s-1} ds. \tag{6.3}$$

We put

$$\tilde{g}_+(u,v,w,z;x)$$
$$= \frac{1}{2\pi i}\cos\left(\tfrac12\pi(v-z)\right)$$
$$\times \int_{(\eta_1)} \Gamma(1+s-v-w)\Gamma(1+s-w-z)$$
$$\times g^*(s,w)(x/2)^{u+v+w+z-2s-1}ds,$$
$$\tilde{g}_-(u,v,w,z;x)$$
$$= -\frac{1}{2\pi i}\int_{(\eta_1)} \cos\left(\tfrac12\pi(v+2w+z-2s)\right)$$
$$\times \Gamma(1+s-v-w)\Gamma(1+s-w-z)$$
$$\times g^*(s,w)(x/2)^{u+v+w+z-2s-1}ds, \tag{6.4}$$

and

$$Y_\pm(u,v,w,z;g;d/c;m,n)$$
$$= \sum_{(k,d)=1} \frac{1}{ck\sqrt{d}} S(m,\pm\bar{d}n;ck)$$
$$\times \tilde{g}_\pm\left(u,v,w,z;\frac{4\pi\sqrt{mn}}{ck\sqrt{d}}\right)$$
$$= \sum_{(k,d)=1} \frac{1}{ck\sqrt{d}} S(n,\pm\bar{d}m;ck)$$
$$\times \tilde{g}_\pm\left(u,v,w,z;\frac{4\pi\sqrt{mn}}{ck\sqrt{d}}\right). \tag{6.5}$$

We have

$$J_+^*(u,v,w,z;g;d/c)$$
$$= [K_+ + K_-](u,v,w,z;g;d/c), \tag{6.6}$$

with

$$K_\pm(u,v,w,z;g;d/c)$$
$$= 2\frac{c^u d^{\frac12(u+v-w+z)}}{(2\pi)^{u-w+1}}$$
$$\times \sum_{m,n} m^{\frac12(v+w-u-z-1)} n^{-\frac12(u+v+w+z-1)} \sigma_{w+z-1}(n)$$
$$\times Y_\pm(u,v,w,z;g;d/c;m,n). \tag{6.7}$$

7. We need to spectrally decompose the sums $Y_\pm$. To this end we shall begin with some basic facts about a generic discrete subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$ and later proceed to the Kuznetsov sum formula for the Hecke congruence subgroup $\Gamma_0(q)$.

Thus, let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which has a fundamental domain of finite volume. We call $\mathfrak{a}$ a cusp of Γ if and only if there exists a $\sigma \in \Gamma$ such that σ is parabolic, i.e., $\mathrm{Tr}(\sigma) = \pm 2$ and $\sigma(\mathfrak{a}) = \mathfrak{a} \in \mathbb{R}\cup\infty$. Let $\Gamma_\mathfrak{a}$ be $\{\sigma \in \Gamma : \sigma(\mathfrak{a}) = \mathfrak{a}\}$, i.e., the stabilizer of $\mathfrak{a}$. Then $\Gamma_\mathfrak{a}$ is cyclic, so all elements in it are parabolic. Hence, there exits a $\sigma_\mathfrak{a}$ such that $\sigma_\mathfrak{a}(\infty) = \mathfrak{a}$ and $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty = [S]$ with $S = \left(\begin{smallmatrix}1&1\\&1\end{smallmatrix}\right)$.

The discussion below depends on the choice of $\sigma_\mathfrak{a}$ which is not unique. If $\sigma'_\mathfrak{a}$ is another choice, then there exists a b such that $\sigma'_\mathfrak{a} = \sigma_\mathfrak{a}S^b$. In fact, since $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = {\sigma'_\mathfrak{a}}^{-1}\Gamma_\mathfrak{a}\sigma'_\mathfrak{a}$, we have $\sigma_\mathfrak{a}S\sigma_\mathfrak{a}^{-1} = \sigma'_\mathfrak{a}S^{\pm1}{\sigma'_\mathfrak{a}}^{-1}$ or $\sigma_\mathfrak{a}^{-1}\sigma'_\mathfrak{a}S^{\pm1} = S\sigma_\mathfrak{a}^{-1}\sigma'_\mathfrak{a}$. On the other hand $\sigma_\mathfrak{a}^{-1}\sigma'_\mathfrak{a}(\infty) = \infty$ implies that $\sigma_\mathfrak{a}^{-1}\sigma'_\mathfrak{a} = \left(\begin{smallmatrix}a&b\\&c\end{smallmatrix}\right)$; and $\left(\begin{smallmatrix}a&b\\&c\end{smallmatrix}\right)\left(\begin{smallmatrix}1&\pm1\\&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&1\\&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\&c\end{smallmatrix}\right)$ yields that $a = \pm c$, that is, $a = c = 1$ and the assertion follows.

Let f be a Γ-automorphic form of weigh $2k$, with a positive integer k; namely, for $\sigma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Gamma$,

$$f(\sigma(z)) = (cz+d)^{2k}f(z)$$
$$= \jmath(\sigma,z)^{2k}f(z). \tag{7.1}$$

The function $f(\sigma_\mathfrak{a}(z))(\jmath(\sigma_\mathfrak{a},z))^{-2k}$ is of period 1. In fact,

$$f(\sigma_\mathfrak{a}S(z))(\jmath(\sigma_\mathfrak{a},S(z)))^{-2k}$$
$$= f(\sigma_\mathfrak{a}S\sigma_\mathfrak{a}^{-1}\sigma_\mathfrak{a}(z))(\jmath(\sigma_\mathfrak{a},S(z)))^{-2k}$$
$$= f(\sigma_\mathfrak{a}(z))(\jmath(\sigma_\mathfrak{a}S\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a}(z)))^{2k}(\jmath(\sigma_\mathfrak{a},S(z)))^{-2k}$$
$$= f(\sigma_\mathfrak{a}(z))\left[\jmath(\sigma_\mathfrak{a}S,z)/\jmath(\sigma_\mathfrak{a},z)\right]^{2k}(\jmath(\sigma_\mathfrak{a},S(z)))^{-2k}$$
$$= f(\sigma_\mathfrak{a}(z))(\jmath(\sigma_\mathfrak{a},z))^{-2k}. \tag{7.2}$$

Thus, if $f(\sigma_\mathfrak{a}(z))$ is regular near ∞, then the function $f(\sigma_\mathfrak{a}(\log z/2\pi i))(\jmath(\sigma_\mathfrak{a},\log z/2\pi i))^{-2k}$ is single valued and regular on a small disk centered and punctured at the origin. Hence

$$f(\sigma_\mathfrak{a}(z))(\jmath(\sigma_\mathfrak{a},z))^{-2k} = \sum_n \varrho(n,\mathfrak{a})\exp(2\pi inz), \tag{7.3}$$

which is called the Fourier expansion of f around the cusp $\mathfrak{a}$.

Note that this expansion depends on the choice of $\sigma_\mathfrak{a}$. In fact If $\sigma'_\mathfrak{a}$ is another choice , then $\sigma'_\mathfrak{a} = \sigma_\mathfrak{a}S^b$ with a b. We have $f(\sigma'_\mathfrak{a}(z))(\jmath(\sigma_{\mathfrak{a}'},z))^{-2k} = f(\sigma_\mathfrak{a}(z+b))(\jmath(\sigma_\mathfrak{a},z+b))^{-2k}$. That is, $\varrho(n,\mathfrak{a})$ is multiplied by $\exp(2\pi inb)$.

If f is regular on the upper half plane $\mathcal{H}=\{z=x+iy:-\infty<x<\infty, y>0\}$ and $\varrho(n,\mathfrak{a})=0$ for any $n\le 0$ and any $\mathfrak{a}$, then f is termed a holomorphic cusp-form. Let $\mathcal{S}_k(\Gamma)$ be the space of all cusp-forms of weight $2k$. Then $\mathcal{S}_k(\Gamma)$ is a finite dimensional Hermitian space with the Petersson inner product

$$\langle f,g\rangle_k=\int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}y^{2k}d\mu(z),\qquad d\mu(z)=dxdy/y^2. \tag{7.4}$$

We let $\{\psi_{j,k}(z), 1\le j\le\vartheta(k)\}$ stand for an orthonormal base of $\mathcal{S}_k(\Gamma)$.

8. Let $k\ge 2$. We introduce the Poincaré series

$$P_m(z,\mathfrak{a};k)=\sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma}(\jmath(\sigma_\mathfrak{a}^{-1}\gamma,z))^{-2k}\exp(2\pi im\sigma_\mathfrak{a}^{-1}\gamma(z)). \tag{8.1}$$

This is a holomorphic cusp form of weight $2k$ for any integer $m>0$. We shall confirm this claim, though we skip the convergence issue, which causes no difficulty when $k\ge 2$.

First, each summand is a function over $\Gamma_\mathfrak{a}\backslash\Gamma$. In fact, if $\Gamma_\mathfrak{a}\gamma=\Gamma_\mathfrak{a}\gamma'$, then $\Gamma_\infty\sigma_\mathfrak{a}^{-1}\gamma=\Gamma_\infty\sigma_\mathfrak{a}^{-1}\gamma'$ and $\sigma_\mathfrak{a}^{-1}\gamma(z)\equiv\sigma_\mathfrak{a}^{-1}\gamma'(z)$ mod 1 as well as $\jmath(\sigma_\mathfrak{a}^{-1}\gamma,z)=\jmath(\sigma_\mathfrak{a}^{-1}\gamma',z)$. Also the relation $P_m(\gamma(z),\mathfrak{a};k)=(\jmath(\gamma,z))^{2k}P_m(z,\mathfrak{a};k)$ is obvious; and $P_m(z,\mathfrak{a};k)$ is regular over $\mathcal{H}$. Thus, it remains to consider the Fourier expansion at a given cusp $\mathfrak{b}$. We have

$$\begin{aligned}&P_m(\sigma_\mathfrak{b}(z),\mathfrak{a};k)(\jmath(\sigma_\mathfrak{b},z))^{-2k}\\&=\sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma}(\jmath(\sigma_\mathfrak{b},z))^{-2k}(\jmath(\sigma_\mathfrak{a}^{-1}\gamma,\sigma_\mathfrak{b}(z)))^{-2k}\\&\quad\times\exp(2\pi im\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}(z))\\&=\sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma}(\jmath(\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b},z))^{-2k}\exp(2\pi im\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}(z))\\&=\sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma}\frac{1}{(cz+d)^{2k}}\exp\left(2\pi im\frac{az+b}{cz+d}\right),\end{aligned} \tag{8.2}$$

where $\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. If $c=0$, then $\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}(\infty)=\infty$ or $\gamma(\mathfrak{b})=\mathfrak{a}$, that is, $\mathfrak{a}\equiv\mathfrak{b}$ mod Γ as well as $\gamma\sigma_\mathfrak{b}=\sigma_\mathfrak{a}S^b$. Moreover, if $\gamma'\sigma_\mathfrak{b}=\sigma_\mathfrak{a}S^{b'}$, then $\gamma'(\mathfrak{b})=\mathfrak{a}=\gamma(\mathfrak{b})$, that is, $\gamma'\gamma^{-1}\in\Gamma_\mathfrak{a}$ or $\Gamma_\mathfrak{a}\gamma=\Gamma_\mathfrak{a}\gamma'$. Hence

$$\sum_{\substack{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma\\c=0}}=\delta_{\mathfrak{a},\mathfrak{b}}\exp(2\pi im(z+b)). \tag{8.3}$$

As to the remaining part, we have

$$\sum_{\substack{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma\\c\ne0}}=\sum_{\substack{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma\\c\ne0}}\frac{1}{(cz+d)^{2k}}\times\exp\left(2\pi im\frac{a}{c}-2\pi im\frac{1}{c(cz+d)}\right). \tag{8.4}$$

We observe that if $\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ appears in the right side, then $\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}S^n\sigma_\mathfrak{b}^{-1}\sigma_\mathfrak{b}=\left(\begin{smallmatrix}a&b+an\\c&d+cn\end{smallmatrix}\right)$ does for all $n\in\mathbb{Z}$. In fact, $\gamma\sigma_\mathfrak{b}S^n\sigma_\mathfrak{b}^{-1}\in\Gamma$ and thus $\Gamma_\mathfrak{a}\gamma\sigma_\mathfrak{b}S^n\sigma_\mathfrak{b}^{-1}$ is an element of $\Gamma_\mathfrak{a}\backslash\Gamma$. Moreover, if $\Gamma_\mathfrak{a}\gamma\sigma_\mathfrak{b}S^m\sigma_\mathfrak{b}^{-1}=\Gamma_\mathfrak{a}\gamma\sigma_\mathfrak{b}S^n\sigma_\mathfrak{b}^{-1}$, then $\sigma_\mathfrak{a}\Gamma_\infty\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}S^m\sigma_\mathfrak{b}^{-1}=\sigma_\mathfrak{a}\Gamma_\infty\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}S^n\sigma_\mathfrak{b}^{-1}$ or $\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}S^m=S^l\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{b}S^n$. This means that

$$\begin{pmatrix}a&b+am\\c&d+cm\end{pmatrix}=\begin{pmatrix}a+cl&b+an+(d+cn)l\\c&d+cn\end{pmatrix}; \tag{8.5}$$

and we get $l=0$, $m=n$, which confirms our claim. On the other hand, since $\{\gamma\sigma_\mathfrak{b}S^n\sigma_\mathfrak{b}^{-1}:n\in\mathbb{Z}\}=\gamma\Gamma_\mathfrak{b}$, we should classify the summands in (8.4) according to the double coset decomposition $\Gamma_\mathfrak{a}\backslash\Gamma/\Gamma_\mathfrak{b}$, which naturally we could have introduced already at (8.2).

We have thus

$$\sum_{\substack{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma\\c\ne0}}=\sum_{\substack{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma/\Gamma_\mathfrak{b}\\c\ne0}}\sum_n\frac{1}{(c(z+n)+d)^{2k}}\times\exp\left(2\pi im\frac{a}{c}-2\pi im\frac{1}{c(c(z+n)+d)}\right). \tag{8.6}$$

More explicitly, we have the relation $\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma/\Gamma_\mathfrak{b}$ is equivalent to $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{b}/\Gamma_\infty$. With this one may proceed just in the same way as the case of the full modular group and get

$$\begin{aligned}&P_m(\sigma_\mathfrak{b}(z),\mathfrak{a};k)(\jmath(\sigma_\mathfrak{b},z))^{-2k}=\delta_{\mathfrak{a},\mathfrak{b}}\exp(2\pi im(z+b))\\&+2\pi(-1)^k\sum_{n>0}\left\{\sum_{c>0}\frac{1}{c}S(m,n;c;\mathfrak{a},\mathfrak{b})\right.\\&\left.\times\left(\frac{n}{m}\right)^{k-\frac12}J_{2k-1}\left(4\pi\frac{\sqrt{mn}}{c}\right)\right\}\exp(2\pi inz).\end{aligned} \tag{8.7}$$

Here

$$S(m,n;c;\mathfrak{a},\mathfrak{b})=\sum_\gamma\exp\left(2\pi i(am+dn)/c\right) \tag{8.8}$$

is a Kloosterman sum associated with Γ, where γ runs over the representatives of $\Gamma_\mathfrak{a}\backslash\Gamma/\Gamma_\mathfrak{b}$ with the same c in the sense remarked after (8.6). The expression (8.8) and the constant b in (8.7) depend of course on the choice of $\sigma_\mathfrak{a}$, $\sigma_\mathfrak{b}$.

The last summands are functions on $\Gamma_{\mathfrak{a}}\backslash\Gamma/\Gamma_{\mathfrak{b}}$. In fact, let $\Gamma_{\mathfrak{a}}\gamma\Gamma_{\mathfrak{b}}=\Gamma_{\mathfrak{a}}\gamma'\Gamma_{\mathfrak{b}}$. Then $\sigma_{\mathfrak{a}}\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}\Gamma_\infty\sigma_{\mathfrak{b}}^{-1}\ni\gamma'$ or $\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}=\sigma_{\mathfrak{a}}^{-1}\gamma'\sigma_{\mathfrak{b}}\Gamma_\infty$, which means that there exist two integers l, l' such that $S^l\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\left(\begin{smallmatrix}a'&b'\\c'&d'\end{smallmatrix}\right)S^{l'}$. Hence $c=c'$ and $a\equiv a'$, $d\equiv d'$ mod c. Also, for each $c>0$ there are at most finitely many double cosets having c as the lower-left element; otherwise the convergence would be violated.

On the assumption that there exists a $c_0>0$ such that for any non-zero integers m, n and any pair of cusps $\mathfrak{a},\mathfrak{b}$

$$\sum_c \frac{1}{c^{2k}}|S(m,n;c;\mathfrak{a},\mathfrak{b})|\ll(mn)^{c_0}, \tag{8.9}$$

We have

$$P_m(\sigma_{\mathfrak{b}}(\infty),\mathfrak{a};k)=0, \tag{8.10}$$

implying that P_m is a holomorphic cusp form of weight $2k$.

9. We consider the spectral decomposition

$$\begin{aligned}&\langle P_m(\cdot,\mathfrak{a};k),P_n(\cdot,\mathfrak{b};k)\rangle_k\\ &=\sum_{j=1}^{\vartheta(k)}\langle P_m(\cdot,\mathfrak{a};k),\psi_{j,k}\rangle_k\overline{\langle P_n(\cdot,\mathfrak{b};k),\psi_{j,k}\rangle_k}.\end{aligned} \tag{9.1}$$

The left side is

$$\begin{aligned}&\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma}\int_{\Gamma\backslash\mathcal{H}}P_m(z,\mathfrak{a};k)\\
&\quad\times\overline{(\jmath(\sigma_{\mathfrak{b}}^{-1}\gamma,z))^{-2k}\exp(2\pi in\sigma_{\mathfrak{b}}^{-1}\gamma(z))}y^{2k}d\mu(z)\\
&=\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma}\int_{\sigma_{\mathfrak{b}}^{-1}\gamma(\Gamma\backslash\mathcal{H})}P_m(\gamma^{-1}\sigma_{\mathfrak{b}}(z),\mathfrak{a};k)\\
&\quad\times\overline{(\jmath(\sigma_{\mathfrak{b}}^{-1}\gamma,\gamma^{-1}\sigma_{\mathfrak{b}}(z)))^{-2k}\exp(2\pi inz)}\\
&\quad\times\frac{y^{2k}d\mu(z)}{|\jmath(\gamma^{-1}\sigma_{\mathfrak{b}},z)|^{4k}}\\
&=\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma}\int_{\sigma_{\mathfrak{b}}^{-1}\gamma(\Gamma\backslash\mathcal{H})}P_m(\gamma^{-1}\sigma_{\mathfrak{b}}(z),\mathfrak{a};k)\\
&\quad\times(\jmath(\gamma^{-1}\sigma_{\mathfrak{b}},z)))^{-2k}\exp(-2\pi in\bar z)y^{2k}d\mu(z)\\
&=\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma}\int_{\sigma_{\mathfrak{b}}^{-1}\gamma(\Gamma\backslash\mathcal{H})}P_m(\sigma_{\mathfrak{b}}(z),\mathfrak{a};k)(\jmath(\gamma^{-1},\sigma_{\mathfrak{b}}(z))^{2k}\\
&\quad\times(\jmath(\gamma^{-1}\sigma_{\mathfrak{b}},z)))^{-2k}\exp(-2\pi in\bar z)y^{2k}d\mu(z)\\
&=\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma}\int_{\sigma_{\mathfrak{b}}^{-1}\gamma(\Gamma\backslash\mathcal{H})}P_m(\sigma_{\mathfrak{b}}(z),\mathfrak{a};k)\\
&\quad\times(\jmath(\sigma_{\mathfrak{b}},z)))^{-2k}\exp(-2\pi in\bar z)y^{2k}d\mu(z)\\
&=\int_{\sigma_{\mathfrak{b}}^{-1}\bigcup_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma}\gamma(\Gamma\backslash\mathcal{H})}P_m(\sigma_{\mathfrak{b}}(z),\mathfrak{a};k)\\
&\quad\times(\jmath(\sigma_{\mathfrak{b}},z)))^{-2k}\exp(-2\pi in\bar z)y^{2k}d\mu(z)\\
&=\int_0^\infty\int_0^1 P_m(\sigma_{\mathfrak{b}},\mathfrak{a};k)(\jmath(\sigma_{\mathfrak{b}},z)))^{-2k}\\
&\quad\times\exp(-2\pi in\bar z)y^{2k-2}dxdy\\
&=2\pi\Gamma(2k-1)(4\pi\sqrt{mn})^{1-2k}\\
&\quad\times\Big\{\frac{1}{2\pi}\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,n}\exp(2\pi inb)+(-1)^k\sum_c\frac{1}{c}S(m,n;c;\mathfrak{a},\mathfrak{b})\\
&\quad\times J_{2k-1}\left(4\pi\frac{\sqrt{mn}}{c}\right)\Big\},\end{aligned} \tag{9.2}$$

where we have used that $\sigma_{\mathfrak{b}}^{-1}\bigcup_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma}\gamma(\Gamma\backslash\mathcal{H})=\sigma_{\mathfrak{b}}^{-1}(\Gamma_{\mathfrak{b}}\backslash\mathcal{H})=\Gamma_\infty\backslash\mathcal{H}$; in fact, since $\sigma_{\mathfrak{b}}\Gamma_\infty\sigma_{\mathfrak{b}}^{-1}(\Gamma_{\mathfrak{b}}\backslash\mathcal{H})=\mathcal{H}$, we have $\Gamma_\infty\sigma_{\mathfrak{b}}^{-1}(\Gamma_{\mathfrak{b}}\backslash\mathcal{H})=\mathcal{H}$.

On the other hand, we have in much the same way

$$\begin{aligned}&\langle P_m(\cdot,\mathfrak{a};k),\psi_{j,k}\rangle_k\\
&=\int_0^\infty\int_0^1\exp(2\pi imz)\overline{\psi_{j,k}(\sigma_{\mathfrak{a}}(z))\jmath(\sigma_{\mathfrak{a}},z)^{-2k}}d\mu(z)\\
&=\Gamma(2k-1)(4\pi m)^{1-2k}\overline{\varrho_{j,k}(m,\mathfrak{a})},\end{aligned} \tag{9.3}$$

where we have put, following (7.3),

$$\begin{aligned}&\psi_{j,k}(\sigma_{\mathfrak{a}}(z))\jmath(\sigma_{\mathfrak{a}},z)^{-2k}\\
&\quad=\sum_{n>0}\varrho_{j,k}(n,\mathfrak{a})\exp(2\pi inz).\end{aligned} \tag{9.4}$$

Hence we have obtained the Petersson Formula:

Lemma 1. *For $k\ge 2$*

$$\begin{aligned}&\frac{1}{2\pi}\frac{\Gamma(2k-1)}{(4\pi\sqrt{mn})^{2k-1}}\sum_{j=1}^{\vartheta(k)}\overline{\varrho_{j,k}(m,\mathfrak{a})}\varrho_{j,k}(n,\mathfrak{b})\\
&=\frac{1}{2\pi}\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,n}\exp(2\pi inb)\\
&+(-1)^k\sum_c\frac{1}{c}S(m,n;c;\mathfrak{a},\mathfrak{b})J_{2k-1}\left(4\pi\frac{\sqrt{mn}}{c}\right),\end{aligned} \tag{9.5}$$

provided Γ satisfies (8.9).

The case $k=1$ can also be treated in much the same way as is done with the full modular group (see [11, pp. 52–54]), excepting that (8.9) should be replaced by the assumption that *there be a constant $\tau<2$ such that for any non-zero integers m, n and for any pair of cusps $\mathfrak{a},\mathfrak{b}$*

$$\sum_c\frac{1}{c^\tau}|S(m,n;c;\mathfrak{a},\mathfrak{b})|\ll|mn|^{c_0}. \tag{9.6}$$

On this the assertion (9.5) holds for all $k\ge1$.

10. We turn to real analytic cusp forms. The procedure is similar to the holomorphic case and also to the full modular situation, and we can be brief.

Let f be a real analytic cusp form of weight zero with respect to Γ so that $f(\gamma(z)) = f(z)$ for all $\gamma \in \Gamma$, and $\Delta f = \nu f$ with $\Delta = -y^2(\partial_x^2 + \partial_y^2)$. Since $f(\sigma_{\mathfrak{a}}(z))$ is of period one, we have the Fourier expansion

$$f(\sigma_{\mathfrak{a}}(z)) = \sum_n \varrho(n, \mathfrak{a}; y) \exp(2\pi i n x). \tag{10.1}$$

We require that

$$\begin{gathered}\lim_{z\to\infty} f(\sigma_{\mathfrak{a}}(z)) = 0 \text{ for any } \mathfrak{a}, \text{ and} \\ \int_{\Gamma\backslash\mathcal{H}} |f|^2 d\mu(z) < \infty.\end{gathered} \tag{10.2}$$

We have then

$$\begin{aligned} &f(\sigma_{\mathfrak{a}}(z)) \\ &= y^{\frac12} \sum_{n\neq 0} \varrho(n,\mathfrak{a}) K_{i\kappa}(2\pi|n|y) \exp(2\pi i n x), \end{aligned} \tag{10.3}$$

where $\nu = \kappa^2 + \frac14$.

One may consider more generally the decomposition of the space $L^2(\Gamma\backslash G)$, $G = \mathrm{PSL}(2,\mathbb{R})$ into irreducible subspaces and appeal to the theory of representations of the Lie group G. This will allow us to deal with all cusp forms of various weights in a unified fashion. However, here we shall rather follow the argument due to Kuznetsov and others.

Thus, let us introduce the Poincaré series of the Selberg type

$$\begin{aligned} &U_m(z,\mathfrak{a};s) \\ &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \left(\mathrm{Im}\sigma_{\mathfrak{a}}^{-1}\gamma(z)\right)^s \exp(2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma(z)), \end{aligned} \tag{10.4}$$

and the Eisenstein series $E(z,\mathfrak{a};s) = U_0(z;\mathfrak{a};s)$, associated with the cusp $\mathfrak{a}$. Arguing as in Section 8, we have the Fourier expansion

$$\begin{aligned} &U_m(\sigma_{\mathfrak{b}}(z),\mathfrak{a};s) = \delta_{\mathfrak{a},\mathfrak{b}} y^s \exp(2\pi i m(z+b)) \\ &+ y^{1-s} \sum_n \exp(2\pi i n x) \sum_c \frac{1}{c^{2s}} S(m,n;c;\mathfrak{a},\mathfrak{b}) \\ &\times \int_{-\infty}^{\infty} \exp\left(-2\pi i n y\xi - \frac{2\pi m}{c^2 y(1-i\xi)}\right) \frac{d\xi}{(1+\xi^2)^s}. \end{aligned} \tag{10.5}$$

On the assumption (9.6), $U_m(\sigma_{\mathfrak{b}}(z),\mathfrak{a};s)$ is regular for $\mathrm{Re}\, s > \tau/2$, and also $U_m(\sigma_{\mathfrak{b}}(z),\mathfrak{a};s) \ll y^{1-\mathrm{Re}\, s}$ as $y \to \infty$. In particular, $U_m(z,\mathfrak{a};s) \in L^2(\Gamma\backslash\mathcal{H})$ if $\mathrm{Re}\, s > \tau/2$. Also we have

$$\begin{aligned} &E(\sigma_{\mathfrak{b}}(z),\mathfrak{a};s) \\ &= \delta_{\mathfrak{a},\mathfrak{b}} y^s + \sqrt{\pi} y^{1-s} \frac{\Gamma\left(s-\frac12\right)}{\Gamma(s)} e_0(s;\mathfrak{a},\mathfrak{b}) \\ &+ \frac{2\pi^s}{\Gamma(s)} y^{\frac12} \sum_{n\neq 0} |n|^{s-\frac12} e_n(s;\mathfrak{a},\mathfrak{b}) \\ &\times K_{s-\frac12}(2\pi|n|y) \exp(2\pi i n x), \end{aligned} \tag{10.6}$$

with

$$e_n(s;\mathfrak{a},\mathfrak{b}) = \sum_c \frac{1}{c^{2s}} S(0,n;\mathfrak{a},\mathfrak{b}). \tag{10.7}$$

It can be shown that $E(\sigma_{\mathfrak{b}}(z),\mathfrak{a};s)$ is meromorphic for all s. Moreover, in the case of congruence subgroups, $E(\sigma_{\mathfrak{b}}(z),\mathfrak{a};s)$ is regular for $\mathrm{Re}\, s \ge \frac12$ except for a simple pole at $s = 1$.

Let $\{\psi_j : j \ge 1\}$ be a complete orthonormal base of the cuspidal subspace of $L^2(\Gamma\backslash\mathcal{H})$ such that $\Delta\psi_j = \nu_j\psi_j$ with $\nu_j = \kappa_j^2 + \frac14$, and

$$\begin{aligned} &\psi_j(\sigma_{\mathfrak{a}}(z)) \\ &= y^{\frac12} \sum_{n\neq 0} \varrho_j(n,\mathfrak{a}) K_{i\kappa_j}(2\pi|n|y) \exp(2\pi i n x). \end{aligned} \tag{10.8}$$

We put also $\psi_0 \equiv (\text{volume of } \Gamma\backslash\mathcal{H})^{-1/2}$. We suppose that Γ is such that no $E(z,\mathfrak{a};s)$ has poles in the interval $\left(\frac12, 1\right)$. Then we have the spectral expansion: For any pair $f, g \in L^2(\Gamma\backslash\mathcal{H})$, it holds that

$$\begin{aligned} \langle f, g\rangle &= \sum_{j=0}^{\infty} \langle f,\psi_j\rangle \overline{\langle g,\psi_j\rangle} \\ &\quad + \frac{1}{4\pi} \sum_{\mathfrak{c}} \int_{-\infty}^{\infty} \mathcal{E}(r,\mathfrak{c};f)\overline{\mathcal{E}(r,\mathfrak{c};g)} dr, \end{aligned} \tag{10.9}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_0$ and

$$\mathcal{E}(r,\mathfrak{c};f) = \int_{\Gamma\backslash\mathcal{H}} f(z) \overline{E\left(z,\mathfrak{c};\tfrac12 + ir\right)} d\mu(z). \tag{10.10}$$

11. We collect here analogues of Bruggeman's and Kuznetsov's formulas: On the basic assumption (9.6) we have:

Lemma 2. *Uniformly for any $n \neq 0$ and $\mathfrak{a}$,*

$$\begin{aligned} &\sum_{\kappa_j\le K} \frac{|\varrho_j(n,\mathfrak{a})|^2}{\cosh \pi\kappa_j} + \sum_{\mathfrak{c}} \int_{-K}^{K} \left|e_n\left(\tfrac12 + ir;\mathfrak{c},\mathfrak{a}\right)\right|^2 dr \\ &\qquad \ll K^2 + |n|^{c_1}, \end{aligned} \tag{11.1}$$

where c_1 depends on τ, c_0 in (9.6). *In particular, we have the bound*

$$\varrho_j(n,\mathfrak{a}) \ll (\kappa_j + |n|^{\frac12 c_1}) \exp\left(\tfrac12\pi\kappa_j\right). \tag{11.2}$$

Lemma 3. *Let $h(r)$ be even, regular and of fast decay on the strip $|\mathrm{Im} r| < \frac{1}{2}+\eta$ with an $\eta > 0$. Then it holds that for any $m, n > 0$ and $\mathfrak{a}, \mathfrak{b}$*

$$\begin{aligned}&\sum_{j=1}^{\infty} \frac{\overline{\varrho_j(m,\mathfrak{a})}\varrho_j(\pm n,\mathfrak{b})}{\cosh \pi\kappa_j} h(\kappa_j)\\ &\quad+\frac{1}{\pi}\sum_{\mathfrak{c}}\int_{-\infty}^{\infty}(n/m)^{ir}\\ &\quad\times \overline{e_m\left(\tfrac12+ir;\mathfrak{c},\mathfrak{a}\right)}e_n\left(\tfrac12+ir;\mathfrak{c},\mathfrak{b}\right)h(r)dr\\ &=\frac{1}{\pi^2}\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,\pm n}\exp(2\pi imb)\int_{-\infty}^{\infty} r\tanh(\pi r)h(r)dr\\ &\quad+\sum_{c}\frac{1}{c}S(m,\pm n;c;\mathfrak{a},\mathfrak{b})h_{\pm}\left(4\pi\frac{\sqrt{mn}}{c}\right),\end{aligned} \tag{11.3}$$

where $\mathfrak{c}$ runs over all inequivalent cusps, and

$$\begin{aligned}h_+(x)&=\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{rh(r)}{\cosh\pi r}J_{2ir}(x)dr,\\ h_-(x)&=\frac{4}{\pi^2}\int_{-\infty}^{\infty} rh(r)\sinh(\pi r)K_{2ir}(x)dr.\end{aligned} \tag{11.4}$$

Lemma 4. *Let φ be smooth and of fast decay over the positive real axis. Then we have, for any $m, n > 0$ and $\mathfrak{a}, \mathfrak{b}$,*

$$\begin{aligned}&\sum_{c}\frac{1}{c}S(m,\pm n;c;\mathfrak{a},\mathfrak{b})\varphi\left(4\pi\frac{\sqrt{mn}}{c}\right)\\ &=\sum_{j=1}^{\infty}\frac{\overline{\varrho_j(m,\mathfrak{a})}\varrho_j(\pm n,\mathfrak{b})}{\cosh\pi\kappa_j}\hat\varphi_{\pm}(\kappa_j)\\ &\quad+\frac{1\pm1}{4\pi(4\pi\sqrt{mn})^{2k-1}}\sum_{k=1}^{\infty}\Gamma(2k)\hat\varphi_+\left(\left(\tfrac12-2k\right)i\right)\\ &\quad\times\sum_{j=1}^{\vartheta(k)}\overline{\varrho_{j,k}(m,\mathfrak{a})}\varrho_{j,k}(n,\mathfrak{b})\\ &\quad+\frac{1}{\pi}\sum_{\mathfrak{c}}\int_{-\infty}^{\infty}(n/m)^{ir}\\ &\quad\times\overline{e_m\left(\tfrac12+ir;\mathfrak{c},\mathfrak{a}\right)}e_n\left(\tfrac12+ir;\mathfrak{c},\mathfrak{b}\right)\hat\varphi_{\pm}(r)dr,\end{aligned} \tag{11.5}$$

where

$$\begin{aligned}\hat\varphi_+(r)&=\frac{\pi i}{2\sinh\pi r}\int_0^{\infty}\left\{J_{2ir}(x)-J_{-2ir}(x)\right\}\varphi(x)\frac{dx}{x},\\ \hat\varphi_-(r)&=2\cosh(\pi r)\int_0^{\infty}K_{2ir}(x)\varphi(x)\frac{dx}{x}.\end{aligned} \tag{11.6}$$

12. With this, we shall consider the specialization $\Gamma = \Gamma_0(q)$. Our discussion overlaps, to a certain extent, with that developed in [3]; however, the present work can be read independently of it. In this section we shall fix a representative set of all cusps inequivalent mod $\Gamma_0(q)$.

We introduce $V = \left\{\left(\begin{smallmatrix}1 & \\ n & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\right\}$ the stabilizer of the point 0 in $\Gamma_0(1)$ and the double coset decomposition

$$\Gamma_0(1) = \bigcup_{\mathfrak{a}} \Gamma_0(q)\gamma_{\mathfrak{a}}V, \tag{12.1}$$

where the symbol $\mathfrak{a}$ is to be regarded temporarily as to be just a label. We begin with a particular $\gamma_{\mathfrak{a}}$, and transform it to a matrix suitable for our purpose. We thus look into the product

$$\begin{aligned}&\begin{pmatrix}a & b\\ cq & d\end{pmatrix}\begin{pmatrix}e & f\\ g & h\end{pmatrix}\begin{pmatrix}1 & \\ n & 1\end{pmatrix}\\ &=\begin{pmatrix}* & *\\ * & k\end{pmatrix}\begin{pmatrix}1 & \\ n & 1\end{pmatrix},\end{aligned} \tag{12.2}$$

where the middle matrix on the left side corresponds to $\gamma_{\mathfrak{a}}$. It is to be observed that g is fixed mod h, because of the action of V. We assume that $h \neq 0$. We have $k = cfq + dh$, and we claim that this can be made equal to (q,h). In fact $c(fq/(q,h)) + d(h/(q,h)) = 1$ is soluble in c and d, for $(fq,h) = (q,h)$; then $d \equiv \overline{h/(q,h)}$ mod $fq/(q,h)$, and d can be a prime large enough so that $(d,q) = 1$, and thus $(d,cq) = 1$. With such a d we may choose a, b to satisfy $ad - bcq = 1$, which confirms our claim. On the other hand, if $h = 0$, then it suffices to put $c = \mathrm{sgn}(f)$, $d = 1$. Thus we may suppose that $\gamma_{\mathfrak{a}} = \left(\begin{smallmatrix}* & *\\ * & w\end{smallmatrix}\right)$ with $w|q$; that is, each coset in (12.1) contains elements of this property.

We then apply (12.1) to the point 0, getting

$$\mathbb{Q}\cup\{\infty\} = \bigcup_{\mathfrak{a}}\Gamma_0(q)\gamma_{\mathfrak{a}}(0). \tag{12.3}$$

This means that $\{\gamma_{\mathfrak{a}}(0) : \mathfrak{a}\}$, with the current definition of $\mathfrak{a}$, is the full set of inequivalent cusps mod $\Gamma_0(q)$. In fact, that $\Gamma_0(q)\gamma_{\mathfrak{a}}(0) \ni \gamma_{\mathfrak{a}'}(0)$ implies readily that $\Gamma_0(q)\gamma_{\mathfrak{a}}V = \Gamma_0(q)\gamma_{\mathfrak{a}'}V$; and the stabilizer in $\Gamma_0(q)$ of $\gamma_{\mathfrak{a}}(0)$ is $\gamma_{\mathfrak{a}}V_{q/w}\gamma_{\mathfrak{a}}^{-1}$ with $V_d = \left\{\left(\begin{smallmatrix}1 & \\ n & 1\end{smallmatrix}\right) : d|n\right\}$, provided $\gamma_{\mathfrak{a}} = \left(\begin{smallmatrix}* & *\\ * & w\end{smallmatrix}\right)$. The labels $\{\mathfrak{a}\}$ indeed coincide with their former designation. Also, it should be noted that the element w is unique to each double coset, which can be proved by considering the relation $\Gamma_0(q)\left(\begin{smallmatrix}* & *\\ * & w\end{smallmatrix}\right)V = \Gamma_0(q)\left(\begin{smallmatrix}* & *\\ * & w'\end{smallmatrix}\right)V$ with respect to either mod w or mod w', getting $w|w'$ and $w'|w$, respectively. Namely, if $w \neq w'$, then $\Gamma_0(q)\left(\begin{smallmatrix}* & *\\ * & w\end{smallmatrix}\right)V \cap \Gamma_0(q)\left(\begin{smallmatrix}* & *\\ * & w'\end{smallmatrix}\right)V = \emptyset$.

Hence, it remains to see when the relation

$$\Gamma_0(q)\begin{pmatrix} e & f \\ g & w\end{pmatrix} V = \Gamma_0(q)\begin{pmatrix} e' & f' \\ g' & w\end{pmatrix} V \tag{12.4}$$

holds, where the two matrices are in $\Gamma_0(1)$ with $w|q$ and $(gg', w) = 1$. We have

$$\begin{aligned}
&\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} e & f \\ g & w\end{pmatrix} = \begin{pmatrix} e' & f' \\ g' & w\end{pmatrix}\begin{pmatrix} 1 & \\ n & 1\end{pmatrix} \quad \text{with } q|c \\
\iff\quad & \begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} e'+nf' & f' \\ g'+nw & w\end{pmatrix}\begin{pmatrix} w & -f \\ -g & e\end{pmatrix} \\
\iff\quad & c = w(g'+nw) - gw = w(g'-g+nw) \\
\iff\quad & w(g'-g+nw) \equiv 0 \bmod q \\
\iff\quad & g'-g+nw \equiv 0 \bmod q/w \\
\iff\quad & g' \equiv g \bmod (w, q/w).
\end{aligned} \tag{12.5}$$

Hence

$$\begin{aligned}
&(12.4) \iff \\
&(gg', w) = 1 \text{ and } g \equiv g' \bmod (w, q/w).
\end{aligned} \tag{12.6}$$

Namely, when $\gamma_{\mathfrak{a}}$ varies with w fixed, then g and thus f runs over the complete residue classes mod $(w, q/w)$ while satisfying $(w, f) = 1$. If $(u, (w, q/w)) = 1$, then obviously there exists an f such that $u \equiv f \bmod (w, q/w)$ and $(w, f) = 1$.

Collecting the above, we have

Lemma 5. *A complete representative set of cusps inequivalent* mod $\Gamma_0(q)$ *is given by*

$$\left\{\frac{u}{w} : w|q,\ (u,w)=1,\ u \bmod (w, q/w)\right\}, \tag{12.7}$$

whose cardinality is

$$\sum_{w|q} \varphi((w, q/w)). \tag{12.8}$$

13. Let us fix the stabilizers of those cusps given in (12.7). To this end we note first that if $\mathfrak{a} \neq \infty$ is a cusp of a discrete group Γ, then

$$\Gamma_{\mathfrak{a}} = \Gamma \cap \left\{\begin{pmatrix} 1+\nu\mathfrak{a} & -\nu\mathfrak{a}^2 \\ \nu & 1-\nu\mathfrak{a}\end{pmatrix} : \nu \in \mathbb{R}\right\}. \tag{13.1}$$

In fact, since $(a\mathfrak{a}+b)/(c\mathfrak{a}+d) = \mathfrak{a}$, $a+d=2$, we see that $\mathfrak{a} = (1-d)/c$, and the assertion follows with $c = \nu$. If $\mathfrak{a} = u/w$ with $w|q$, $(u,w)=1$, then

$$\begin{aligned}
&\Gamma_{u/w} \\
&= \Gamma_0(q) \cap \left\{\begin{pmatrix} 1+\nu\dfrac{u}{w} & -\nu\dfrac{u^2}{w^2} \\ \nu & 1-\nu\dfrac{u}{w}\end{pmatrix} : \nu \in \mathbb{R}\right\},
\end{aligned} \tag{13.2}$$

and thus $\nu \in \mathbb{Z}$, $\nu \equiv 0 \bmod q$, $\nu \equiv 0 \bmod w^2$; namely

$$\Gamma_{u/w} = \left\{\begin{pmatrix} 1+\nu\dfrac{u}{w} & -\nu\dfrac{u^2}{w^2} \\ \nu & 1-\nu\dfrac{u}{w}\end{pmatrix}\right\}, \tag{13.3}$$

with $\mathbb{Z} \ni \nu \equiv 0 \bmod [w^2, q]$.

We write

$$\begin{aligned}
q &= vw = (v,w)^2 v^* w^*, \\
v^* &= \frac{v}{(v,w)},\ w^* = \frac{w}{(v,w)}.
\end{aligned} \tag{13.4}$$

We put

$$\varpi_{u/w} = \begin{pmatrix} u & \dfrac{u\overline{u}-1}{w} \\ w & \overline{u}\end{pmatrix}, \quad u\overline{u} \equiv 1 \bmod w, \tag{13.5}$$

and

$$\tau_{v^*} = \begin{pmatrix} \sqrt{v^*} & \\ & \dfrac{1}{\sqrt{v^*}}\end{pmatrix}. \tag{13.6}$$

Obviously we have $\varpi_{u/w}(\infty) = u/w$. Moreover, we have

$$\begin{aligned}
&\varpi_{u/w}^{-1}\begin{pmatrix} 1+\nu\dfrac{u}{w} & -\nu\dfrac{u^2}{w^2} \\ \nu & 1-\nu\dfrac{u}{w}\end{pmatrix}\varpi_{u/w} \\
&= \begin{pmatrix} \overline{u} & -\dfrac{u\overline{u}-1}{w} \\ -w & u\end{pmatrix}\begin{pmatrix} 1+\nu\dfrac{u}{w} & -\nu\dfrac{u^2}{w^2} \\ \nu & 1-\nu\dfrac{u}{w}\end{pmatrix} \\
&\quad \times \begin{pmatrix} u & \dfrac{u\overline{u}-1}{w} \\ w & \overline{u}\end{pmatrix} \\
&= \begin{pmatrix} 1 & -\dfrac{\nu}{w^2} \\ & 1\end{pmatrix} \\
&= \tau_{v^*}\begin{pmatrix} 1 & -\dfrac{\nu}{v^* w^2} \\ & 1\end{pmatrix}\tau_{v^*}^{-1}.
\end{aligned} \tag{13.7}$$

Hence, on noting that $[w^2, q] = v^* w^2$, we get

$$\Gamma_{u/w} = \varpi_{u/w}\tau_{v^*}\Gamma_\infty\tau_{v^*}^{-1}\varpi_{u/w}^{-1}, \tag{13.8}$$

which is equivalent to

$$\Gamma_{u/w} = \varpi_{u/w}\left[S^{v^*}\right]\varpi_{u/w}^{-1}. \tag{13.9}$$

14. In the the special instance where $q = v_i w_i$ with $(v_i, w_i) = 1$, we shall consider the structure of the double coset decomposition $\Gamma_{1/w_1}\backslash\Gamma_0(q)/\Gamma_{1/w_2}$ and associated Kloosterman sums.

To this end we put

$$\begin{aligned}\sigma_{1/w_i} &= \varpi_{1/w_i}\tau_{v_i}S^{-\overline{w}_i/v_i}\\ &= \varpi_{1/w_i}S^{-\overline{w}_i}\tau_{v_i},\end{aligned} \tag{14.1}$$

where $w_i\overline{w}_i \equiv 1 \bmod v_i$. The choice of a particular value of $\overline{w}_i$ is irrelevant to our discussion of the Kloosterman sums, as we shall show later. Note that

$$\Gamma_{1/w_i} = \sigma_{1/w_i}\Gamma_\infty\sigma_{1/w_i}^{-1}, \tag{14.2}$$

as is implied by (13.8).

We shall prove that

$$\begin{aligned}&S^{\overline{w}_1}\varpi_{1/w_1}^{-1}\Gamma_0(q)\varpi_{1/w_2}S^{-\overline{w}_2}\\ &=\left\{\begin{pmatrix}(v_1,w_2)k & (v_1,v_2)l\\ (w_1,w_2)r & (w_1,v_2)s\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z})\right\}\end{aligned} \tag{14.3}$$

with $k,l,r,s\in\mathbb{Z}$ (cf. [6, p. 534]; note that there q is square-free but here not assumed to be so). In fact, we have, by (13.5),

$$\varpi_{1/w_i}S^{-\overline{w}_i} = \begin{pmatrix}1 & -\overline{w}_i\\ w_i & 1-w_i\overline{w}_i\end{pmatrix}; \tag{14.4}$$

thus for $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(q)$

$$\begin{aligned}&S^{\overline{w}_1}\varpi_{1/w_1}^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\varpi_{1/w_2}S^{-\overline{w}_2}\\ &\equiv\begin{pmatrix}0 & *\\ * & *\end{pmatrix}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\\ &=\begin{pmatrix}0 & *\\ * & *\end{pmatrix}\bmod (v_1,w_2),\\ &\equiv\begin{pmatrix}0 & *\\ * & *\end{pmatrix}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}* & *\\ * & 0\end{pmatrix}\\ &=\begin{pmatrix}* & 0\\ * & *\end{pmatrix}\bmod (v_1,v_2),\\ &\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\\ &=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\bmod (w_1,w_2),\\ &\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}* & *\\ * & 0\end{pmatrix}\\ &=\begin{pmatrix}* & *\\ * & 0\end{pmatrix}\bmod (w_1,v_2).\end{aligned} \tag{14.5}$$

On the other hand, we have that

$$\begin{aligned}&\varpi_{1/w_1}S^{-\overline{w}_1}\begin{pmatrix}(v_1,w_2)k & (v_1,v_2)l\\ (w_1,w_2)r & (w_1,v_2)s\end{pmatrix}S^{\overline{w}_2}\varpi_{1/w_2}^{-1}\\ &\equiv\begin{pmatrix}* & *\\ * & 0\end{pmatrix}\begin{pmatrix}0 & *\\ * & *\end{pmatrix}\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\\ &=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\bmod (v_1,w_2),\\ &\equiv\begin{pmatrix}* & *\\ * & 0\end{pmatrix}\begin{pmatrix}* & 0\\ * & *\end{pmatrix}\begin{pmatrix}0 & *\\ * & *\end{pmatrix}\\ &=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\bmod (v_1,v_2),\\ &\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\\ &=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\bmod (w_1,w_2),\\ &\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\begin{pmatrix}* & *\\ * & 0\end{pmatrix}\begin{pmatrix}0 & *\\ * & *\end{pmatrix}\\ &=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\bmod (w_1,v_2),\end{aligned} \tag{14.6}$$

and that $(v_1,w_2)(v_1,v_2)(w_1,w_2)(w_1,v_2) = q$. This proves (14.3).

Hence, we have, with $k,l,r,s\in\mathbb{Z}$,

$$\begin{aligned}&\Gamma_{1/w_1}\backslash\Gamma_0(q)/\Gamma_{1/w_2}\\ &=\sigma_{1/w_1}\Gamma_\infty\sigma_{1/w_1}^{-1}\backslash\Gamma_0(q)/\sigma_{1/w_2}\Gamma_\infty\sigma_{1/w_2}^{-1}\\ &\Longleftrightarrow\\ &\Gamma_\infty\backslash\tau_{v_1}^{-1}S^{\overline{w}_1}\varpi_{1/w_1}^{-1}\Gamma_0(q)\varpi_{1/w_2}S^{-\overline{w}_2}\tau_{v_2}/\Gamma_\infty\\ &\Longleftrightarrow\\ &\Gamma_\infty\backslash\tau_{v_1}^{-1}\left\{\begin{pmatrix}(v_1,w_2)k & (v_1,v_2)l\\ (w_1,w_2)r & (w_1,v_2)s\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})\right\}\tau_{v_2}/\Gamma_\infty\\ &\Longleftrightarrow\\ &\Gamma_\infty\backslash\left\{\begin{pmatrix}(v_1,w_2)k\sqrt{v_2/v_1} & (v_1,v_2)l/\sqrt{v_1v_2}\\ (w_1,w_2)r\sqrt{v_1v_2} & (w_1,v_2)s\sqrt{v_1/v_2}\end{pmatrix}\right\}/\Gamma_\infty\\ &\Longleftrightarrow\end{aligned}$$

classifying the solutions of

$(v_1,w_2)(w_1,v_2)sk-(w_1,w_2)(v_1,v_2)rl = 1$

according to $(v_1,w_2)k\sqrt{v_2/v_1}, (w_1,v_2)s\sqrt{v_1/v_2}$

$\mathrm{mod}(w_1,w_2)r\sqrt{v_1v_2}$; note the remark after (8.6)

$\Longleftrightarrow$

the moduli of the Kloosterman sums

have the form $(w_1,w_2)r\sqrt{v_1v_2}$

with $((v_1,w_2)(w_1,v_2),r) = 1$ and

$(v_1,w_2)(w_1,v_2)sk\equiv 1 \bmod (w_1,w_2)(v_1,v_2)r$

$(v_1,w_2)k \bmod v_1(w_1,w_2)r$

$$\begin{aligned}
&\longleftrightarrow k \bmod (v_1,v_2)(w_1,w_2)r\\
&(w_1,v_2)s \bmod v_2(w_1,w_2)r\\
&\longleftrightarrow s \bmod (v_1,v_2)(w_1,w_2)r\\
&\Longleftrightarrow\\
&c=(w_1,w_2)r\sqrt{v_1v_2},\ ((v_1,w_2)(w_1,v_2),r)=1,\\
&S(m,n;c;1/w_1,1/w_2)\\
&=\sum \exp\left(\frac{2\pi i(km+ns)}{(v_1,v_2)(w_1,w_2)r}\right)\\
&=S(\overline{(v_1,w_2)}m,\overline{(w_1,v_2)}n;(v_1,v_2)(w_1,w_2)r), \qquad (14.7)
\end{aligned}$$

where the last sum is over s, k mod $(v_1,v_2)(w_1,w_2)r$, $(v_1,w_2)(w_1,v_2)sk \equiv 1 \bmod (v_1,v_2)(w_1,w_2)r$, and the last member is an ordinary Kloosterman sum.

It remains to show the irrelevance of the choice of values of $\overline{w}_j$. In fact, if we replace $\overline{w}_j$ by $\overline{w}_j+nv_j$, $n\in\mathbb{Z}$, then the first equivalence assertion in (14.7) does not change, for we have $\tau_{v_j}^{-1}S^{nv_j}\tau_{v_j}=S^n\in\Gamma_\infty$.

In particular, we find that if $q=cd$, $(c,d)=1$, and $(r,d)=1$, then

$$\begin{aligned}
&S(m,n;cr\sqrt{d};1/q,1/c)\\
&=S(m,n;cr\sqrt{d};\infty,1/c)\\
&=S(m,\overline{d}n;cr), \qquad (14.8)
\end{aligned}$$

on the specification (14.1) of $\sigma_{1/q}$ and $\sigma_{1/c}$.

15. We still need to see if (9.6) is satisfied by the generic $\Gamma_0(q)$. Until very recently we had been unable to locate any rigorous treatment of those generalized Kloosterman sums over $\Gamma_0(q)$ in literature, excepting [9] and [10] where the case with q square-free is explicitly discussed on the basis of (14.7). With this situation, R.W. Bruggeman kindly provided us with a treatment [1] of the sums using a partly adelic reasoning; and it is assured that (9.6) indeed holds with any $\Gamma_0(q)$. Here we shall prove the same with an alternative elementary method; this section can be read independently of [1].

We shall first redefine the Kloosterman sums associated with the two cusps u_i/w_i, $i=1,2$, which are in the set (12.7), by introducing the convention

$$\sigma_{u_i/w_i}=\varpi_{u_i/w_i}\tau_{v_i^*}, \qquad (15.1)$$

with v_i^* as in Section 13, which is effective within this section only. Note that when $u_i=1$ this does not coincide with (14.1); when discussing the absolute values of generalized Kloosterman sums, obviously no difference is caused. Also, it is expedient to use the Bruhat decomposition; that is, in the big cell of $\mathrm{PSL}(2,\mathbb{R})$ we have

$$\begin{aligned}
\begin{pmatrix} a & b\\ c & d\end{pmatrix} &= \begin{pmatrix} 1 & a/c\\ & 1\end{pmatrix}\begin{pmatrix} & -1/c\\ c & \end{pmatrix}\begin{pmatrix} 1 & d/c\\ & 1\end{pmatrix}\\
&=B[a,d;c], \qquad (15.2)
\end{aligned}$$

say.

With this, let $\varkappa_q$ be the characteristic function of the set $\Gamma_0(q)\subset\mathrm{PSL}(2,\mathbb{R})$. Then Kloosterman sums associated with the two cusps u_i/w_i, $i=1,2$, have moduli $c\sqrt{v_1^*v_2^*}$, $c\in\mathbb{N}$; and under (15.1) we have that

$$\begin{aligned}
&S(m,n;c\sqrt{v_1^*v_2^*};u_1/w_1,u_2/w_2)\\
&=\sum_{\substack{ad\equiv 1 \bmod c\\ a \bmod v_1^*c\\ d \bmod v_2^*c}} \varkappa_q\left(\varpi_{u_1/w_1}B[a,d;c]\varpi_{u_2/w_2}^{-1}\right)\\
&\quad\times\exp\left(2\pi i\left(\frac{ma}{v_1^*c}+\frac{nd}{v_2^*c}\right)\right), \qquad (15.3)
\end{aligned}$$

where $a,c,d\in\mathbb{Z}$. In fact, by (13.8) we need to consider the double coset decomposition

$$\begin{aligned}
&\Gamma_\infty\backslash\tau_{v_1^*}^{-1}\varpi_{u_1/w_1}^{-1}\Gamma_0(q)\varpi_{u_2/w_2}\tau_{v_2^*}/\Gamma_\infty\\
&=\Gamma_\infty\backslash\tau_{v_1^*}^{-1}\Big\{B[a,d;c]:\\
&\qquad\varkappa_q(\varpi_{u_1/w_1}B[a,d,c]\varpi_{u_2/w_2}^{-1})=1\Big\}\tau_{v_2^*}/\Gamma_\infty\\
&=\Gamma_\infty\backslash\Big\{B\left[a\sqrt{v_2^*/v_1^*},d\sqrt{v_1^*/v_2^*};c\sqrt{v_1^*v_2^*}\right]:\\
&\qquad\varkappa_q(\varpi_{u_1/w_1}^{-1}B[a,d,c]\varpi_{u_2/w_2})=1\Big\}/\Gamma_\infty, \quad (15.4)
\end{aligned}$$

where $B[a,d;c]\in\Gamma_0(1)$, since $\varpi_{u_1/w_1}^{-1}\Gamma_0(q)\varpi_{u_2/w_2}\subset\Gamma_0(1)$. The expression (15.3) readily follows. In passing, we note that

$$|S(m,n;c\sqrt{v_1^*v_2^*};u_1/w_1,u_2/w_2)|\le v_1^*v_2^*\varphi(c), \quad (15.5)$$

for the number of summands on the right of (15.3) is less than or equal to $v_1^*v_2^*\varphi(c)$. In fact, a unique d mod c corresponds to each a, $(a,c)=1$, or v_2^* classes d mod v_2^*c to each of $v_1^*\varphi(c)$ classes a mod v_1^*c with $(a,c)=1$.

We remark that $\varkappa_q\left(\varpi_{u_1/w_1}B[a,d;c]\varpi_{u_2/w_2}^{-1}\right)$ is a function over a mod v_1^*c and d mod v_2^*c. To see this, we use the relation

$$\begin{aligned}
&\varpi_{u_1/w_1}B[a+a',d+d';c]\varpi_{u_2/w_2}^{-1}\\
&=\varpi_{u_1/w_1}\begin{pmatrix} 1 & a'/c\\ & 1\end{pmatrix}\varpi_{u_1/w_1}^{-1}\cdot\varpi_{u_1/w_1}B[a,d;c]\\
&\times\varpi_{u_2/w_2}^{-1}\cdot\varpi_{u_2/w_2}\begin{pmatrix} 1 & d'/c\\ & 1\end{pmatrix}\varpi_{u_2/w_2}^{-1}; \qquad (15.6)
\end{aligned}$$

and (13.9) gives that

$$\varpi_{u_1/w_1}\begin{pmatrix}1 & a'/c\\ & 1\end{pmatrix}\varpi_{u_1/w_1}^{-1}\in\Gamma_{u_1/w_1}\subset\Gamma_0(q),$$
$$\varpi_{u_2/w_2}\begin{pmatrix}1 & d'/c\\ & 1\end{pmatrix}\varpi_{u_2/w_2}^{-1}\in\Gamma_{u_2/w_2}\subset\Gamma_0(q),\quad(15.7)$$

provided $v_1^*|(a'/c)\in\mathbb{Z}$, $v_2^*|(d'/c)\in\mathbb{Z}$, which proves the assertion.

Next, we shall show that if $ad\equiv 1 \bmod c$, then

$$\varkappa_q\left(\varpi_{u_1/w_1}B[a,d;c]\varpi_{u_2/w_2}^{-1}\right)$$
$$=\varkappa_q\left(\varpi_{\overline{c^*}u_1/w_1}B[a,d;c_0]\varpi_{\overline{c^*}u_2/w_2}^{-1}\right),\quad(15.8)$$

where $c=c_0c^*$ with $c_0=(c,q^\infty)$, and $\overline{c^*}c^*\equiv 1 \bmod q$; note that $\overline{c^*}u_i/w_i$ are cusps of $\Gamma_0(q)$. In fact, computing the lower-left element of $\varpi_{u_1/w_1}B[a,d;c]\varpi_{u_2/w_2}^{-1}$, we see that the value of the left side of (15.8) equals 1 if and only if

$$\overline{u}_2(aw_1+c\overline{u}_1)$$
$$\equiv w_2\left(w_1(ad-1)/c+d\overline{u}_1\right) \bmod q;\quad(15.9)$$

and this is equivalent to the congruence

$$\overline{\overline{c^*}u_2}(aw_1+c_0\overline{\overline{c^*}u_1})$$
$$\equiv w_2\left(w_1(ad-1)/c_0+d\overline{\overline{c^*}u_1}\right) \bmod q,\quad(15.10)$$

which immediately implies (15.8).

Hence we have

$$S(m,n;c\sqrt{v_1^*v_2^*};u_1/w_1,u_2/w_2)$$
$$=\sum_{\substack{ad\equiv 1 \bmod c\\ a \bmod v_1^*c\\ d \bmod v_2^*c}}\varkappa_q\left(\varpi_{\overline{c^*}u_1/w_1}B[a,d;c_0]\varpi_{\overline{c^*}u_2/w_2}^{-1}\right)$$
$$\times\exp\left(2\pi i\left(\frac{ma}{v_1^*c}+\frac{nd}{v_2^*c}\right)\right).\quad(15.11)$$

Here we have

$$\frac{1}{v_i^*c}\equiv\frac{\widetilde{c_i^*}}{v_i^*c_0}+\frac{\widetilde{v_i^*c_0}}{c^*} \bmod 1,\quad(15.12)$$

with $\widetilde{c_i^*}c^*\equiv 1 \bmod v_ic_0$, $\widetilde{v_i^*c_0}v_i^*c_0\equiv 1 \bmod c^*$. Inserting this into (15.11), putting $a\equiv a_0 \bmod v_1^*c_0$, $a\equiv a^* \bmod c^*$, $d\equiv d_0 \bmod v_1^*c_0$, $d\equiv d^* \bmod c^*$, and further, noting the congruence property of $\varkappa_q$ proved in (15.6)–(15.7), we may write (15.11) as

$$S(m,n;c\sqrt{v_1^*v_2^*};u_1/w_1,u_2/w_2)$$
$$=\sum_{\substack{ad\equiv 1 \bmod c\\ a \bmod v_1^*c\\ d \bmod v_2^*c}}\varkappa_q\left(\varpi_{\overline{c^*}u_1/w_1}B[a_0,d_0;c_0]\varpi_{\overline{c^*}u_2/w_2}^{-1}\right)$$
$$\times\exp\left(2\pi i\left(\frac{\widetilde{c_1^*}ma_0}{v_1^*c_0}+\frac{\widetilde{c_2^*}nd_0}{v_2^*c_0}\right)\right)$$
$$\times\exp\left(2\pi i\left(\frac{\widetilde{v_1^*c_0}ma^*}{c^*}+\frac{\widetilde{v_2^*c_0}nd^*}{c^*}\right)\right).\quad(15.13)$$

We have thus obtained the factorization

$$S(m,n;c\sqrt{v_1^*v_2^*};u_1/w_1,u_2/w_2)$$
$$=S(\widetilde{c_1^*}m,\widetilde{c_2^*}n;c_0\sqrt{v_1^*v_2^*};\overline{c^*}u_1/w_1,\overline{c^*}u_2/w_2)$$
$$\times S(\widetilde{v_1^*c_0}m,\widetilde{v_2^*c_0}n;c^*),\quad(15.14)$$

where the last S-factor is an ordinary Kloosterman sum.

In particular, applying (15.5) and the Weil bound, respectively, to the first and the second factors on the right side of (15.14), we get

$$|S(m,n;c\sqrt{v_1^*v_2^*};u_1/w_1,u_2/w_2)|$$
$$\le v_1^*v_2^*\varphi(c_0)|S(\widetilde{v_1^*c_0}m,\widetilde{v_2^*c_0}n;c^*)|$$
$$\ll v_1^*v_2^*c_0((m,n,c^*)c^*)^{\frac12+\varepsilon},\quad(15.15)$$

with the implied constant depending only on ε. Thus we have, for any $\xi>\frac12$,

$$\sum_c\frac{1}{(c\sqrt{v_1^*v_2^*})^\tau}|S(m,n;c\sqrt{v_1^*v_2^*};u_1/w_1,u_2/w_2)|$$
$$\ll(v_1^*v_2^*)^{1-\frac12\tau}\left(\sum_{c|q^\infty}\frac{1}{c^{\tau-1}}\right)\left(\sum_c\frac{(m,n,c)^\xi}{c^{\tau-\xi}}\right),\quad(15.16)$$

which is finite if $\tau-\xi>1$. Therefore, we have proved that any $\Gamma_0(q)$ satisfies (9.6) with $\tau>\frac32$.

REMARK 3. The methods in [2] and [12] extend to $M_2(g;A)$ with an arbitrary A. Since they are independent of any non-trivial treatment of generalized Kloosterman sums, the above confirmation of (9.6) for generic $\Gamma_0(q)$ could be regarded as redundant, as far as the spectral decomposition of $M_2(g;A)$ is concerned.

16. With this, we return to the second line of (6.5). We stress that hereafter we shall again work with the definition (14.1).

In view of (14.8) we have

$$Y_\pm(u,v,w,z;g;d/c;m,n)$$
$$=\sum_{(k,d)=1}\frac{1}{ck\sqrt{d}}S(n,\pm m;ck\sqrt{d};\infty,1/c)$$
$$\times\tilde g_\pm\left(u,v,w,z;4\pi\frac{\sqrt{mn}}{ck\sqrt{d}}\right).\quad(16.1)$$

Thus Lemma 4 gives the expansion

$$
\begin{aligned}
&Y_\pm(u,v,w,z;g;d/c;m,n))\\
&=\sum_{j=1}^\infty [g]_\pm(\kappa_j;u,v,w,z)\frac{\overline{\varrho_j(n,\infty)}\varrho_j(\pm m,1/c)}{\cosh\pi\kappa_j}\\
&+\frac{1\pm1}{4\pi(4\pi\sqrt{mn})^{2k-1}}\sum_{k=1}^\infty \Gamma(2k)[g]_+\big(\left(\tfrac12-2k\right)i;u,v,w,z\big)\\
&\times\sum_{j=1}^{\vartheta(k)}\overline{\varrho_{j,k}(n,\infty)}\varrho_{j,k}(m,1/c)\\
&+\frac1\pi\sum_{\mathfrak c}\int_{-\infty}^\infty [g]_\pm(r;u,v,w,z)(m/n)^{ir}\\
&\times\overline{e_n\left(\tfrac12+ir;\mathfrak c,\infty\right)}e_m\left(\tfrac12+ir;\mathfrak c,1/c\right)dr, \qquad (16.2)
\end{aligned}
$$

where

$$
\begin{aligned}
&[g]_+(r;u,v,w,z)\\
&=\frac{\pi i}{2\sinh\pi r}\int_0^\infty \{J_{2ir}(x)-J_{-2ir}(x)\}\,\tilde g_+(u,v,w,z;x)\frac{dx}{x},\\
&[g]_-(r;u,v,w,z)\\
&=2\cosh(\pi r)\int_0^\infty K_{2ir}(x)\tilde g_-(u,v,w,z;x)\frac{dx}{x}. \qquad (16.3)
\end{aligned}
$$

Further, by (6.6)–(6.7) we have that

$$
\begin{aligned}
&\frac{(2\pi)^{u-w+1}}{2c^u d^{\frac12(u+v-w+z)}}J_+^*(u,v,w,z;g;d/c)\\
&=\sum_\pm\sum_{j=1}^\infty\frac{[g]_\pm(\kappa_j;u,v,w,z)}{\cosh\pi\kappa_j}\\
&\quad\times\left\{\sum_m\frac{\overline{\varrho_j(n,\infty)}\sigma_{w+z-1}(n)}{n^{\frac12(u+v+w+z-1)}}\right\}\\
&\quad\times\left\{\sum_n\frac{\varrho_j(\pm n,1/c)}{n^{\frac12(u-v-w+z+1)}}\right\}\\
&+\frac1{2\pi}\sum_{k=1}^\infty\frac{(2k-1)!}{(4\pi)^{2k-1}}[g]_+\big(\left(\tfrac12-2k\right)i;u,v,w,z\big)\\
&\quad\times\left\{\sum_m\frac{\overline{\varrho_{j,k}(n,\infty)}\sigma_{w+z-1}(n)}{n^{k-\frac12}n^{\frac12(u+v+w+z-1)}}\right\}\\
&\quad\times\left\{\sum_n\frac{\varrho_{j,k}(\pm n,1/c)}{n^{k-\frac12}n^{\frac12(u-v-w+z+1)}}\right\}\\
&+\frac1\pi\sum_\pm\sum_{\mathfrak c}\int_{-\infty}^\infty[g]_\pm(r;u,v,w,z)\\
&\quad\times\left\{\sum_m\frac{\overline{e_n(\frac12+ir;\mathfrak c,\infty)}\sigma_{w+z-1}(n)}{n^{\frac12(u+v+w+z-1)+ir}}\right\}\\
&\quad\times\left\{\sum_n\frac{e_n(\frac12+ir;\mathfrak c,1/c)}{n^{\frac12(u-v-w+z+1)-ir}}\right\}dr, \qquad (16.4)
\end{aligned}
$$

as Lemma 2 and the rapid decay of $[g]_\pm(r;u,v,w,z)$ yield absolute convergence on the right side, provided (4.1) (see [11, Section 4.5]).

17. We need to continue (16.4) to a neighborhood of the point $(u,v,w,z)=\left(\frac12,\frac12,\frac12,\frac12\right)$. The continuation of $[g]_\pm$ is known already ([11, Section 4.6]), and we are concerned with the nature of L-functions:

$$
\begin{aligned}
L_j^\pm(s;1/c)&=\sum_n\varrho_j(\pm n,1/c)n^{-s},\\
D_j(s,\alpha)&=\sum_n\overline{\varrho_j(n,\infty)}\sigma_\alpha(n)n^{-s},\\
L_{j,k}(s;1/c)&=\sum_n\varrho_{j,k}(n,1/c)n^{-s-k+\frac12},\\
D_{j,k}(s,\alpha)&=\sum_n\overline{\varrho_{j,k}(n,\infty)}\sigma_\alpha(n)n^{-s-k+\frac12}, \qquad (17.1)
\end{aligned}
$$

where the sums converge absolutely if $\mathrm{Re}\,s$ is sufficiently large, because of (11.2). We shall especially require uniform bounds for these functions. The Dirichlet series involved in the last integral are to be discussed in detail later, but under the restriction on A mentioned in the introduction.

In our continuation procedure of the right side of (16.4), we exploit the fact that above L-functions admit meromorphic continuation to $\mathbb{C}$ with respect to s, and with respect to α as well in the second and the fourth L-functions. To reach (16.4) we appealed to Lemma 4, and hence the bound (9.6) becomes crucial. Moreover, the contribution of the continuous spectrum in (16.4) makes it clear how important for us to have explicit representation of Fourier coefficients of Eisenstein series at each cusp, and this is of course closely related to the structure of generalized Kloosterman sums which is partly discussed in Section 15.

We begin with relations between $\sigma_{\mathfrak a}$ defined by (14.1) and the two basic involutions $J: z\mapsto-\overline z$, and $F_q: z\mapsto -1/qz$, which satisfy

$$
J\varGamma_0(q)J^{-1}=\varGamma_0(q),\quad F_q\varGamma_0(q)F_q^{-1}=\varGamma_0(q). \qquad (17.2)
$$

We have

$$
\begin{gathered}
J\sigma_{\mathfrak a}=\gamma_1\sigma_{\mathfrak b_1}S^{b_1},\quad F_q\sigma_{\mathfrak a}=\gamma_2\sigma_{\mathfrak b_2}S^{b_2},\\
\gamma_1,\gamma_2\in\varGamma_0(q),\ b_1,b_2\in\mathbb{R},
\end{gathered} \qquad (17.3)
$$

where $J(\mathfrak a)$, $F_q(\mathfrak a)$ are equivalent to $\mathfrak b_1$, $\mathfrak b_2$, respectively. For instance, the latter identity is due to the fact that the stabilizer of $\mathfrak b_2$ is

$$
\begin{aligned}
&(\gamma_2^{-1}F_q\sigma_{\mathfrak a})\varGamma_\infty(\gamma_2^{-1}F_q\sigma_{\mathfrak a})^{-1}\\
&=\gamma_2^{-1}F_q\varGamma_{\mathfrak a}F_q^{-1}\gamma_2\subset\varGamma_0(q) \qquad (17.4)
\end{aligned}
$$

(see the remark made prior to (7.1)).

The reflection operator J is isometric over $L^2(\Gamma\backslash\mathcal{H})$, for $J(\Gamma\backslash\mathcal{H})$ is a fundamental domain, and

$$\begin{aligned}\|\psi J\|^2 &= \int_{\Gamma\backslash\mathcal{H}} |\psi J|^2 d\mu \\ &= \int_{J(\Gamma\backslash\mathcal{H})} |\psi|^2 d\mu \\ &= \int_{\Gamma\backslash\mathcal{H}} |\psi|^2 d\mu = \|\psi\|^2. \end{aligned} \tag{17.5}$$

Besides, we have $J\Delta = \Delta J$ as well as the first relation in (17.3). Hence $\psi_j J$ is a cusp form belonging to the same eigenspace as ψ_j, for $\psi_j J(\sigma_{\mathfrak{a}}(z)) = \psi_j(\sigma_{\mathfrak{b}_1}(z + b_1))$ converges to 0 as z tends to ∞. Thus J can be diagonalized on each eigenspace of Δ; that is, we may choose an orthonormal base $\{\psi_j\}$ in such a way that

$$\psi_j(-\overline{z}) = \epsilon_j \psi_j(z), \quad \epsilon_j = \pm 1. \tag{17.6}$$

Also, we observe that

$$\begin{aligned} &J\sigma_{1/c}J\sigma_{1/c}^{-1} \\ &= \begin{pmatrix} \sqrt{d} & -f/\sqrt{d} \\ -c\sqrt{d} & (1+cf)/\sqrt{d} \end{pmatrix} \\ &\times \begin{pmatrix} (1+cf)/\sqrt{d} & -f/\sqrt{d} \\ -c\sqrt{d} & \sqrt{d} \end{pmatrix} \\ &= \begin{pmatrix} 1+2cf & -2f \\ -2c(1+cf) & 1+2cf \end{pmatrix} \in \Gamma_0(cd). \end{aligned} \tag{17.7}$$

This implies that

$$\begin{aligned} &\psi_j(J\sigma_{1/c}J(z)) = \psi_j(\sigma_{1/c}(z)) \\ &\iff \epsilon_j\psi_j(\sigma_{1/c}(-\overline{z})) = \psi_j(\sigma_{1/c}(z)); \end{aligned} \tag{17.8}$$

namely

$$\varrho_j(-n, 1/c) = \epsilon_j \varrho_j(n, 1/c). \tag{17.9}$$

In particular, we have

$$L_j^-(s; 1/c) = \epsilon_j L_j^+(s; 1/c). \tag{17.10}$$

Next, we consider the action of the Fricke operator F_q. We put $F = F_{cd}$. Then each $\psi_j F$ is $\Gamma_0(cd)$- invariant, and is a cusp form such that $\Delta\psi_j F = \nu_j\psi_j F$; in fact it is a unit vector as

$$\begin{aligned} &\int_{\Gamma\backslash\mathcal{H}} |\psi_j F(z)|^2 d\mu(z) \\ &= \int_{F\Gamma\backslash\mathcal{H}} |\psi_j(z)|^2 d\mu(z) \\ &= \int_{\Gamma\backslash\mathcal{H}} |\psi_j(z)|^2 d\mu(z) = 1, \end{aligned} \tag{17.11}$$

for $F\Gamma\backslash\mathcal{H}$ is a fundamental domain of $\Gamma = \Gamma_0(cd)$; moreover, $\psi_j F(\sigma_{\mathfrak{a}}(z)) = \psi_j(\sigma_{\mathfrak{b}_2}(z + b_2))$ converges to 0 as z tends to ∞. Since $FJ = JF$, we may assume, besides (17.6), that

$$\psi_j F = \varpi_j \psi_j, \quad \varpi_j = \pm 1. \tag{17.12}$$

Further, we observe

$$\begin{aligned} &\sigma_{1/c}F\sigma_{1/c}^{-1}F \\ &= \frac{1}{cd}\begin{pmatrix} \sqrt{d} & f/\sqrt{d} \\ c\sqrt{d} & (1+cf)/\sqrt{d} \end{pmatrix}\begin{pmatrix} & -1 \\ cd & \end{pmatrix} \\ &\times \begin{pmatrix} (1+cf)/\sqrt{d} & -f/\sqrt{d} \\ -c\sqrt{d} & \sqrt{d} \end{pmatrix}\begin{pmatrix} & -1 \\ cd & \end{pmatrix} \\ &= \frac{1}{cd}\begin{pmatrix} cf\sqrt{d} & -\sqrt{d} \\ c(1+cf)\sqrt{d} & -c\sqrt{d} \end{pmatrix} \\ &\times \begin{pmatrix} -cf\sqrt{d} & -(1+cf)/\sqrt{d} \\ cd\sqrt{d} & c\sqrt{d} \end{pmatrix} \\ &= \begin{pmatrix} -cf^2 - d & -1 - \dfrac{1+cf}{d} \\ cd\left(\dfrac{f+cf^2}{d} - 1\right) & -c - \dfrac{(1+cf)^2}{d} \end{pmatrix}, \end{aligned} \tag{17.13}$$

which is in $\Gamma_0(cd)$. Hence we have

$$\begin{aligned} \psi_j(\sigma_{1/c}F(z)) &= \psi_j(F^{-1}\sigma_{1/c}(z)) \\ &= \psi_j(F\sigma_{1/c}(z)) \\ &= \varpi_j\psi_j(\sigma_{1/c}(z)); \end{aligned} \tag{17.14}$$

that is, we have

$$\psi_j\left(\sigma_{1/c}\left(-1/cdz\right)\right) = \varpi_j\psi_j\left(\sigma_{1/c}\left(z\right)\right). \tag{17.15}$$

18. We may now prove the functional equation for $L_j(s; 1/c) = L_j^+(s; 1/c)$; note that we have (17.10). We have to discuss two cases separately according as $\epsilon_j = +1$ or -1.

The case $\epsilon_j = +1$: We have, by (17.9),

$$\begin{aligned} &\int_0^\infty \psi_j\left(\sigma_{1/c}\left(\frac{iy}{\sqrt{cd}}\right)\right) y^{s-\frac{3}{2}} dy \\ &= 2\int_0^\infty \left(\frac{y}{\sqrt{cd}}\right)^{\frac{1}{2}} \\ &\times \sum_{n>0} \varrho_j(n, 1/c) K_{i\kappa_j}\left(2\pi\frac{n}{\sqrt{cd}}y\right) y^{s-\frac{3}{2}} dy \\ &= 2^{s-1}(cd)^{-\frac{1}{4}}\left(\frac{2\pi}{\sqrt{cd}}\right)^{-s} \\ &\times \Gamma\left(\tfrac{1}{2}(s+i\kappa_j)\right)\Gamma\left(\tfrac{1}{2}(s-i\kappa_j)\right) L_j(s; 1/c). \end{aligned} \tag{18.1}$$

On the other hand, by (17.15),

$$\begin{aligned}
&\int_0^\infty \psi_j\left(\sigma_{1/c}\left(\frac{iy}{\sqrt{cd}}\right)\right) y^{s-\frac32}dy\\
&=\int_1^\infty \psi_j\left(\sigma_{1/c}\left(\frac{iy}{\sqrt{cd}}\right)\right) y^{s-\frac32}dy\\
&\quad+\int_1^\infty \psi_j\left(\sigma_{1/c}\left(\frac{i}{\sqrt{cd}y}\right)\right) y^{1-s-\frac32}dy\\
&=\int_1^\infty \left\{\psi_j\left(\sigma_{1/c}\left(\frac{iy}{\sqrt{cd}}\right)\right) y^{s-\frac32}\right.\\
&\quad\left.+\varpi_j\psi_j\left(\sigma_{1/c}\left(\frac{iy}{\sqrt{cd}}\right)\right) y^{1-s-\frac32}\right\}dy, \quad (18.2)
\end{aligned}$$

which is entire in s, for $\psi_j\sigma_{1/c}$ decays exponentially as y tends to $+\infty$. Namely, the function $L_j(s;1/c)$ is entire, and we have

$$\begin{aligned}
&\left(\frac{\pi}{\sqrt{cd}}\right)^{-s}\Gamma\left(\tfrac12(s+i\kappa_j)\right)\Gamma\left(\tfrac12(s-i\kappa_j)\right)L_j(s;1/c)\\
&=\varpi_j\left(\frac{\pi}{\sqrt{cd}}\right)^{1-s}\Gamma\left(\tfrac12(1-s+i\kappa_j)\right)\Gamma\left(\tfrac12(1-s-i\kappa_j)\right)\\
&\times L_j(1-s;1/c), \quad (18.3)
\end{aligned}$$

By the duplication formula for Γ-function, one may transform this relation into

$$\begin{aligned}
&L_j(s;1/c)\\
&=\frac{\varpi_j}{\pi}\left(\frac{2\pi}{\sqrt{cd}}\right)^{2s-1}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j)\\
&\times(\cosh\pi\kappa_j-\cos\pi s)\,L_j(1-s;1/c). \quad (18.4)
\end{aligned}$$

The case $\epsilon_j=-1$: We have

$$\begin{aligned}
&\psi_j(\sigma_{1/c}(z))\\
&=2i\sqrt{y}\sum_{n>0}\varrho_j(n,1/c)K_{i\kappa_j}(2\pi ny)\sin(2\pi nx), \quad (18.5)
\end{aligned}$$

We put $f_j(z)=\partial_x\psi_j(\sigma_{1/c}(z-\overline{c}/d))$. We have

$$\begin{aligned}
&f_j(z)\\
&=4\pi i\sqrt{y}\sum_{n>0}n\varrho_j(n,1/c)K_{i\kappa_j}(2\pi ny)\cos(2\pi nx), \quad (18.6)
\end{aligned}$$

which implies that as $x\to0$

$$\psi_j(\sigma_{1/c}(z))=f_j(iy)x+O(x^2) \quad (18.7)$$

as well as

$$\begin{aligned}
&\psi_j\left(\sigma_{1/c}(-1/cdz)\right)\\
&=\psi_j\left(\sigma_{1/c}(i/cdy-x/cdy^2+O(x^2))\right)\\
&=-(x/cdy^2)f_j(i/cdy)+O(x^2); \quad (18.8)
\end{aligned}$$

that is,

$$f_j(i/cdy)=-\varpi_j cdy^2 f_j(iy). \quad (18.9)$$

Hence,

$$\begin{aligned}
&\int_0^\infty f_j\left(\frac{iy}{\sqrt{cd}}\right)y^{s-\frac12}dy=\pi i(cd)^{-\frac14}\left(\frac{\pi}{\sqrt{cd}}\right)^{-s-1}\\
&\times\Gamma\left(\tfrac12(1+s+i\kappa_j)\right)\Gamma\left(\tfrac12(1+s-i\kappa_j)\right)\\
&\times L_j(s;1/c); \quad (18.10)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty f_j\left(\frac{iy}{\sqrt{cd}}\right)y^{s-\frac12}dy=\int_1^\infty\left\{f_j\left(\frac{iy}{\sqrt{cd}}\right)y^{s-\frac12}\right.\\
&\quad\left.-\varpi_j f_j\left(\frac{iy}{\sqrt{cd}}\right)y^{1-s-\frac12}\right\}dy. \quad (18.11)
\end{aligned}$$

Namely, we have that

$$\begin{aligned}
L_j(s;1/c)&=-\frac{\varpi_j}{\pi}\left(\frac{2\pi}{\sqrt{cd}}\right)^{2s-1}\\
&\times\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j)\\
&\times(\cosh\pi\kappa_j+\cos\pi s)\,L_j(1-s;1/c). \quad (18.12)
\end{aligned}$$

Lemma 6. *The function $L_j(s;1/c)$ is entire, and it holds that for any s*

$$\begin{aligned}
L_j(s;1/c)&=\frac{\varpi_j}{\pi}\left(\frac{2\pi}{\sqrt{cd}}\right)^{2s-1}\\
&\times\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j)\\
&\times(\epsilon_j\cosh\pi\kappa_j-\cos\pi s)\,L_j(1-s;1/c). \quad (18.13)
\end{aligned}$$

We have also

$$L_j(s;1/c)\ll(\kappa_j+|s|+1))^{c_0}\exp\left(\tfrac12\pi\kappa_j\right), \quad (18.14)$$

where the constant c_0 depends at most on $\operatorname{Re}s$*, and the implied constant on* $\operatorname{Re}s$.

The second assertion follows via a convexity argument.

We may omit the discussion on $L_{j,k}$, as it is analogous to L_j.

19. We turn to $D_j(s,\alpha)$. There are at least two possible ways for us to take here. One is to exploit the theory of Hecke operators in order to relate D_j with a product of two values of Hecke L-functions analogously as we did in the case of $M_2(g;1)$ in [11]. However, the cusp form ψ_j cannot generally be assumed to be such that the corresponding Hecke series is fully decomposed into an Euler product. This is because those $\varrho_j(n,\infty)$ with

$n|(cd)^\infty$ are not well related to eigenvalues of Hecke operators, and thus the corresponding part of $D_j(s,\alpha)$ causes difficulties in the continuation as well as the estimation procedures, which is a serious drawback of the method as far as our present purpose is concerned. One may appeal to the notion of new forms whose Hecke series admits a full Euler product; yet it does not seem to resolve our difficulties. Hence, we shall take the second method which is in fact a special instance of applications of Rankin's unfolding method (see [11, pp. 181–182]). This causes, however, still a technical difficulty, for it requires us to have an explicit description of the scattering matrix of $\Gamma_0(q)$ and all Fourier coefficients of Eisenstein series at each cusp (see (24.1) below). This task is highly involved. The note [1] contains, in fact, a discussion of the arithmetical nature of those Fourier coefficients and the result appears to be essentially adequate for our purpose, if we let our reasoning in the later sections be somewhat inexplicit; note that the same can be done by extending (15.14) to a full localization. Under such a circumstance, it may be appropriate for us to make here a compromise by introducing the assumption that A is defined by a sum over square-free integers, as underlined in the introduction. Since we have (14.7), this eases our task considerably, yet it does not seem to restrict the scope of our method. In the future, we shall work out a fuller account of $M_2(d;A)$.

20. Thus, we shall hereafter assume that

$$q = cd \text{ is square-free.} \tag{20.1}$$

By Lemma 5 in Section 12, we have now

$$\{\text{inequivalent cusps of } \Gamma_0(q)\} \equiv \left\{\frac{1}{w} : w|q\right\}; \tag{20.2}$$

and we have (14.7) for any combination of cusps. In particular, for those Hecke congruence groups that are relevant in the sequel, (9.6) and thus Lemmas 2–4 have been verified, without the discussion in Section 15.

To make Lemmas 3–4 more explicit, let us compute the Fourier coefficients of Eisenstein series at each cusp. Thus, by the assertion (14.7),

$$\begin{aligned} E(\sigma_{1/w_2}(z), 1/w_1; s) &= \delta_{w_1,w_2} y^s \\ &+ \sqrt{\pi} y^{1-s} \frac{\Gamma\left(s-\frac12\right)}{\Gamma(s)} \\ &\times \sum_{((v_1,w_2)(w_1,v_2),r)=1} \frac{\varphi((v_1,v_2)(w_1,w_2)r)}{((w_1,w_2)r\sqrt{v_1v_2})^{2s}} \\ &+ 2\sqrt{y}\frac{\pi^s}{\Gamma(s)} \sum_{n\neq 0} \exp(2\pi i n x) K_{s-\frac12}(2\pi|n|y)|n|^{s-\frac12} \\ &\times \sum_{((v_1,w_2)(w_1,v_2),r)=1} \frac{c_{(v_1,v_2)(w_1,w_2)r}(n)}{((w_1,w_2)r\sqrt{v_1v_2})^{2s}}, \end{aligned} \tag{20.3}$$

where the last numerator is a Ramanujan sum. We have

$$\begin{aligned} &\sum_{((v_1,w_2)(w_1,v_2),r)=1} \frac{\varphi((v_1,v_2)(w_1,w_2)r)}{((w_1,w_2)r\sqrt{v_1v_2})^{2s}} \\ &= \frac{1}{(w_1,w_2)^{2s}(v_1,v_2)^s} \left\{ \sum_{(r,q)=1} \frac{\varphi(r)}{r^{2s}} \right\} \\ &\times \left\{ \sum_{r|((v_1,v_2)(w_1,w_2))^\infty} \frac{\varphi((v_1,v_2)(w_1,w_2)r)}{r^{2s}} \right\} \\ &= \frac{1}{(w_1,w_2)^{2s}(v_1,v_2)^s} \prod_{p|(v_1,v_2)} \left(\sum_{j=0}^\infty \frac{\varphi(p^{j+1})}{p^{2js}} \right) \\ &\times \prod_{p|(w_1,w_2)} \left(\sum_{j=0}^\infty \frac{\varphi(p^{j+1})}{p^{2js}} \right) \left\{ \sum_{(r,q)=1} \frac{\varphi(r)}{r^{2s}} \right\} \\ &= \frac{\zeta(2s-1)}{\zeta(2s)} \prod_{p|(v_1,v_2)(w_1,w_2)} \left(\frac{p-1}{p^{2s}-1} \right) \\ &\times \prod_{p|(v_1,w_2)(w_1,v_2)} \left(\frac{p^s-p^{1-s}}{p^{2s}-1} \right). \end{aligned} \tag{20.4}$$

Next,

$$\begin{aligned} &\sum_{((v_1,w_2)(w_1,v_2),r)=1} \frac{c_{(v_1,v_2)(w_1,w_2)r}(n)}{((w_1,w_2)r\sqrt{v_1v_2})^{2s}} \\ &= \frac{1}{(w_1,w_2)^{2s}(v_1v_2)^s} \left\{ \sum_{(r,q)=1} \frac{c_r(n)}{r^{2s}} \right\} \\ &\times \left\{ \sum_{r|((v_1,v_2)(w_1,w_2))^\infty} \frac{c_{(v_1,v_2)(w_1,w_2)r}(n)}{r^{2s}} \right\}. \end{aligned} \tag{20.5}$$

We have

$$\begin{aligned} &\sum_{r|((v_1,v_2)(w_1,w_2))^\infty} \frac{c_{(v_1,v_2)(w_1,w_2)r}(n)}{r^{2s}} \\ &= \prod_{p|(v_1,v_2)(w_1,w_2)} \left\{ \sum_{j=0}^\infty \frac{c_{p^{j+1}}(n)}{p^{2js}} \right\} \\ &= \prod_{p|(v_1,v_2)(w_1,w_2)} p^{2s} \left\{ \sum_{j=0}^\infty \frac{c_{p^j}(n)}{p^{2js}} - 1 \right\} \\ &= ((v_1,v_2)(w_1,w_2))^{2s} \\ &\times \prod_{p|(v_1,v_2)(w_1,w_2)} \left\{ \sigma_{1-2s}(n_p)\left(1-\frac{1}{p^{2s}}\right) - 1 \right\} \end{aligned} \tag{20.6}$$

and

$$\sum_{(r,q)=1} \frac{c_r(n)}{r^{2s}} = \frac{\sigma_{1-2s}(n,\chi_q)}{L(2s,\chi_q)}, \tag{20.7}$$

where $n_p = (n, p^\infty)$ and χ_q is the principal character mod$q$. Thus,

$$\sum_{((v_1,w_2)(w_1,v_2),r)=1} \frac{c_{(v_1,v_2)(w_1,w_2)r}(n)}{((w_1,w_2)r\sqrt{v_1v_2})^{2s}}$$
$$= \frac{\sigma_{1-2s}(n,\chi_q)}{L(2s,\chi_q)} \left(\frac{(v_1,v_2)}{[v_1,v_2]}\right)^s$$
$$\times \prod_{p|(v_1,v_2)(w_1,w_2)} \left\{\sigma_{1-2s}(n_p)\left(1-\frac{1}{p^{2s}}\right)-1\right\}. \tag{20.8}$$

Collecting these assertions, we obtain in particular that

Lemma 7. *The function* $s(1-s)\Gamma(s)L(2s,\chi_{cd}) \times E(\sigma_{1/w_2}(z), 1/w_1; s)$ *is regular for all s, and it is* $\ll y^{\mathrm{Re}\, s} + y^{1-\mathrm{Re}\, s}$ *as* $y = \mathrm{Re}\, z$ *tends to infinity, as far as s remains bounded.*

21. Lemma 2 holds safely for $\Gamma = \Gamma_0(q)$, $\mu(q) \neq 0$, and Lemmas 3 and 4 become as follows (see [10]):

Lemma 8. *Let* $h(r)$ *be even, regular and of fast decay on the strip* $|\mathrm{Im} r| < \frac12 + \eta$ *with an* $\eta > 0$. *Then it holds that for any* $m, n > 0$ *and* $w_1|q, w_2|q$

$$\sum_{j=1}^\infty \frac{\overline{\varrho_j(m,1/w_1)}\varrho_j(\pm n,1/w_2)}{\cosh \pi\kappa_j} h(\kappa_j)$$
$$+\frac{1}{\pi}\sum_{q=vw}\int_{-\infty}^\infty \left(\frac{n}{m}\right)^{ir} \frac{\sigma_{2ir}(m;\chi_q)\sigma_{-2ir}(n;\chi_q)}{|L(1+2ir,\chi_q)|^2}$$
$$\times \left(\frac{(v,v_1)}{[v,v_1]}\right)^{\frac12 - ir} \left(\frac{(v,v_2)}{[v,v_2]}\right)^{\frac12 + ir}$$
$$\times \prod_{p|(v,v_1)(w,w_1)} \left\{\sigma_{2ir}(m_p)\left(1-\frac{1}{p^{1-2ir}}\right)-1\right\}$$
$$\times \prod_{p|(v,v_2)(w,w_2)} \left\{\sigma_{-2ir}(n_p)\left(1-\frac{1}{p^{1+2ir}}\right)-1\right\} h(r)dr$$
$$=\frac{1}{\pi^2}\delta_{w_1,w_2}\delta_{m,\pm n}\exp(2\pi i m b_{w_1,w_2})$$
$$\times \int_{-\infty}^\infty r\tanh(\pi r)h(r)dr$$
$$+ \sum_{(r,\,(v_1,w_2)(w_1,v_2))=1} \frac{1}{(w_1,w_2)r\sqrt{v_1v_2}}$$
$$\times S(\overline{(v_1,w_2)}m, \pm\overline{(w_1,v_2)}n; (v_1,v_2)(w_1,w_2)r)$$
$$\times h_\pm\left(\frac{4\pi\sqrt{mn}}{(w_1,w_2)r\sqrt{v_1v_2}}\right), \tag{21.1}$$

with $h_\pm$ *as in* (11.4).

Lemma 9. *Let* φ *be smooth and of fast decay over the positive real axis. Then we have, for any* $m, n > 0$ *and* $w_1|q, w_2|q$,

$$\sum \frac{S(\overline{(v_1,w_2)}m, \pm\overline{(w_1,v_2)}n; (v_1,v_2)(w_1,w_2)r)}{(w_1,w_2)r\sqrt{v_1v_2}}$$
$$\times \varphi\left(\frac{4\pi\sqrt{mn}}{(w_1,w_2)r\sqrt{v_1v_2}}\right)$$
$$=\sum_{j=1}^\infty \frac{\overline{\varrho_j(m,1/w_1)}\varrho_j(\pm n,1/w_2)}{\cosh\pi\kappa_j}\hat\varphi_\pm(\kappa_j)$$
$$+\frac{1\pm1}{4\pi(4\pi\sqrt{mn})^{2k-1}}\sum_{k=1}^\infty \Gamma(2k)\hat\varphi_+\left(\left(\tfrac12 - 2k\right)i\right)$$
$$\times \sum_{j=1}^{\vartheta(k)} \overline{\varrho_{j,k}(m,1/w_1)}\varrho_{j,k}(n,1/w_2)$$
$$+\frac{1}{\pi}\sum_{q=vw}\int_{-\infty}^\infty \left(\frac{n}{m}\right)^{ir}\frac{\sigma_{2ir}(m;\chi_q)\sigma_{-2ir}(n;\chi_q)}{|L(1+2ir,\chi_q)|^2}$$
$$\times \left(\frac{(v,v_1)}{[v,v_1]}\right)^{\frac12 - ir}\left(\frac{(v,v_2)}{[v,v_2]}\right)^{\frac12+ir}$$
$$\times \prod_{p|(v,v_1)(w,w_1)}\left\{\sigma_{2ir}(m_p)\left(1-\frac{1}{p^{1-2ir}}\right)-1\right\}$$
$$\times \prod_{p|(v,v_2)(w,w_2)}\left\{\sigma_{-2ir}(n_p)\left(1-\frac{1}{p^{1+2ir}}\right)-1\right\}$$
$$\times \hat\varphi_\pm(r)dr, \tag{21.2}$$

where $(r, (v_1,w_2)(w_1,v_2)) = 1$ *in the first sum, and the transforms* $\hat\varphi_\pm$ *are as in* (11.6).

We specialize the last assertion as in (14.8), and have, in place of (16.4),

$$\frac{(2\pi)^{u-w+1}}{2c^u d^{\frac12(u+v-w+z)}} J_+^*(u,v,w,z;g;d/c)$$
$$=\sum_\pm \sum_{j=1}^\infty \frac{[g]_\pm(\kappa_j;u,v,w,z)}{\cosh\pi\kappa_j}$$
$$\times\left\{\sum_n \frac{\overline{\varrho_j(n,\infty)}\sigma_{w+z-1}(n)}{n^{\frac12(u+v+w+z-1)}}\right\}$$
$$\times\left\{\sum_n \frac{\varrho_j(\pm n,1/c)}{n^{\frac12(u-v-w+z+1)}}\right\}$$
$$+2\sum_{k=1}^\infty \frac{(2k-1)!}{(4\pi)^{2k}}[g]_+\left(\left(\tfrac12 - 2k\right)i;u,v,w,z\right)$$
$$\times\left\{\sum_n \frac{\overline{\varrho_{j,k}(n,\infty)}\sigma_{w+z-1}(n)}{n^{k-\frac12}n^{\frac12(u+v+w+z-1)}}\right\}$$
$$\times\left\{\sum_n \frac{\varrho_{j,k}(n,1/c)}{n^{k-\frac12}n^{\frac12(u-v-w+z+1)}}\right\}$$
$$+\frac{1}{\pi d^{\frac12+ir}}\sum_\pm\sum_{cd=c_1d_1}\frac{1}{d_1}\int_{-\infty}^\infty \frac{(d_1,d)^{1+2ir}}{|L(1+2ir,\chi_{cd})|^2}$$

$$\times [g]_{\pm}(r;u,v,w,z)$$
$$\times \sum_n \frac{\sigma_{2ir}(n;\chi_{cd})\sigma_{w+z-1}(n)}{n^{\frac12(u+v+w+z-1)+ir}}$$
$$\times \prod_{p|c_1}\left\{\sigma_{2ir}(n_p)\left(1-\frac{1}{p^{1-2ir}}\right)-1\right\}$$
$$\times \sum_n \frac{\sigma_{-2ir}(n;\chi_{cd})}{n^{\frac12(u-v-w+z+1)-ir}}$$
$$\times \prod_p \left\{\sigma_{-2ir}(n_p)\left(1-\frac{1}{p^{1+2ir}}\right)-1\right\}dr, \quad (21.3)$$

with $q=cd$, where $p|(c_1,c)(d_1,d)$ in the last product; and we have used the fact that $\sigma_{1/cd}\in\Gamma_0(cd)$ and thus $\varrho_j(n,1/cd)=\varrho_j(n,\infty)$, $\varrho_{j,k}(n,1/cd)=\varrho_{j,k}(n,\infty)$.

22. We now deal with the function $D_j(s,\alpha)$. As remarked in Section 19, we shall employ the unfolding method.

To this end we introduce the scattering matrix $\mathbb{S}$ of $\Gamma_0(cd)$. We thus write (20.3) as

$$E(\sigma_{1/w_2}(z),1/w_1;s) \\ =\delta_{w_1,w_2}y^s+\varphi(s;w_1,w_2)y^{1-s}+\cdots. \quad (22.1)$$

We put

$$\mathbb{S}(s)=\Big(\varphi(s;w_1,w_2)\Big)_{w_1,w_2|cd}. \quad (22.2)$$

and

$$\mathbb{E}(s)=\begin{pmatrix}\vdots\\ E(z,1/w;s)\\ \vdots\end{pmatrix}_{w|cd}, \quad (22.3)$$

so that

$$\mathbb{E}(s)=\begin{pmatrix}1&&\\&\ddots&\\&&1\end{pmatrix}y^s+\mathbb{S}(s)y^{1-s}+\cdots, \quad (22.4)$$

where the error terms decays exponentially and is $O(y^{\frac12-\varepsilon})$ as y tends to infinity and to 0, respectively.

We have the functional equation

$$\mathbb{E}(z,s)=\mathbb{S}(s)\mathbb{E}(z,1-s), \quad (22.5)$$

provided both sides are finite. To confirm this, we let $\mathrm{Re}\, s$, Im s be sufficiently large. Then (22.4) implies in particular that $\mathbb{E}(z,1-s)-\mathbb{S}(1-s)\mathbb{E}(z,s)$ is in an obvious vector extension of $L^2(\Gamma_0(q)\backslash\mathcal{H})$. However, this vector function, if not trivial, has the eigenvalue $s(1-s)$ against Δ the hyperbolic Laplacian. Since Δ is selfadjoint, its eigenvalues $s(1-s)$ should be real, which is a contradiction, and hence (22.5) holds for all complex s by analytic continuation as far as $\mathbb{E}(z,s)$ is finite. Consequently, we have got also

$$\mathbb{S}(s)\mathbb{S}(1-s)=1. \quad (22.6)$$

23. We shall assume $\epsilon_j=1$ till the end of Section 24.

Let $E(z,s)$ be the Eisenstein series for $\Gamma_0(1)$, and put $E^*(z,s)=\pi^{-s}\Gamma(s)\zeta(2s)E(z,s)$, so that

$$E^*(z,s)=E^*(z,1-s) \quad (23.1)$$

and

$$E^*(z,s)=\pi^{-s}\Gamma(s)\zeta(2s)y^s \\ +\pi^{s-1}\Gamma(1-s)\zeta(2(1-s))y^{1-s} \\ +2\sqrt{y}\sum_{n\neq0}|n|^{s-\frac12}\sigma_{1-2s}(n) \\ \times K_{s-\frac12}(2\pi|n|y)\exp(2\pi nx), \quad (23.2)$$

which shows that $s(1-s)E^*(z,s)$ is regular for all s. We have, on a suitable assumption on s,α to secure convergence, that

$$\int_{\Gamma_0(cd)\backslash\mathcal{H}}\overline{\psi_j(z)}E^*\left(z,\tfrac12(1-\alpha)\right) \\ \times E\left(z,\infty;s-\tfrac12\alpha\right)d\mu(z) \\ =\int_{\Gamma_\infty\backslash\mathcal{H}}\overline{\psi_j(z)}E^*\left(z,\tfrac12(1-\alpha)\right)y^{s-\frac12\alpha}d\mu(z) \\ =4\sum_{n>0}n^{-\frac12\alpha}\sigma_\alpha(n)\overline{\varrho_j(n,\infty)} \\ \times\int_0^\infty K_{\frac12\alpha}(2\pi ny)K_{i\kappa_j}(2\pi ny)y^{s-\frac12\alpha-1}dy \\ =\frac{\Gamma(s,\alpha;\kappa_j)}{2\pi^{s-\frac12\alpha}\Gamma\left(s-\frac12\alpha\right)}D_j(s,\alpha), \quad (23.3)$$

with

$$\Gamma(s,\alpha;\kappa)=\Gamma\left(\tfrac12(s+i\kappa)\right)\Gamma\left(\tfrac12(s-i\kappa)\right) \\ \times\Gamma\left(\tfrac12(s-\alpha+i\kappa)\right)\Gamma\left(\tfrac12(s-\alpha-i\kappa)\right). \quad (23.4)$$

On noting this, we consider also the relation

$$\int_{\Gamma_0(cd)\backslash\mathcal{H}}\overline{\psi_j(z)}E^*\left(z,\tfrac12(1-\alpha)\right) \\ \times E\left(z,1/w;s-\tfrac12\alpha\right)d\mu(z) \\ =\frac{\Gamma(s,\alpha;\kappa_j)}{2\pi^{s-\frac12\alpha}\Gamma\left(s-\frac12\alpha\right)}D_j(s,\alpha;1/w), \quad (23.5)$$

where

$$D_j(s,\alpha;1/w) = \sum_{n>0} \overline{\varrho_j(n,1/w)}\sigma_\alpha(n,1/w)n^{-s}, \quad (23.6)$$

with $\sigma_\alpha(n,1/w)$ an analogue of $\sigma_\alpha(n)$.

By (13.7), we have $E^*(\sigma_{1/w}(z),s) = E^*(\tau_v(z),s) = E^*(vz,s)$, and thus

$$\begin{aligned} E^*(\sigma_{1/w}(z),s) &= \pi^{-s}\Gamma(s)\zeta(2s)(vy)^s \\ &+ \pi^{s-1}\Gamma(1-s)\zeta(2(1-s))(vy)^{1-s} \\ &+ 2\sqrt{y}\sum_{n\neq 0}|n|^{s-\frac12}\sigma_{1-2s}(n,1/w) \\ &\times K_{s-\frac12}(2\pi|n|y)\exp(2\pi n x), \end{aligned} \quad (23.7)$$

That is, we have

$$\sigma_\alpha(n,1/w) = v^{\frac12(\alpha+1)}\sigma_\alpha(n/v), \quad (23.8)$$

which vanishes if $v \nmid n$.

Put

$$\mathbb{D}_j(s,\alpha) = \begin{pmatrix} \vdots \\ D_j(s,\alpha;1/w) \\ \vdots \end{pmatrix}_{w|cd}. \quad (23.9)$$

Then we have, by (22.5) and (23.1),

$$\begin{aligned} &\frac{\Gamma(s,\alpha;\kappa_j)}{\pi^{s-\frac12\alpha}\Gamma\left(s-\frac12\alpha\right)}\mathbb{D}_j(s,\alpha) \\ &= \frac{\Gamma(1-s,-\alpha;\kappa_j)}{\pi^{1-s+\frac12\alpha}\Gamma\left(1-s+\frac12\alpha\right)}\mathbb{S}\left(s-\tfrac12\alpha\right) \\ &\times \mathbb{D}_j(1-s,-\alpha). \end{aligned} \quad (23.10)$$

In particular, we get the functional equation

$$\begin{aligned} D_j(s,\alpha) &= \pi^{2s-\alpha-1}\frac{\Gamma(1-s,-\alpha;\kappa_j)}{\Gamma(s,\alpha;\kappa_j)}\frac{\Gamma\left(s-\frac12\alpha\right)}{\Gamma\left(1-s+\frac12\alpha\right)} \\ &\times \sum_{w|cd}\varphi\left(s-\tfrac12\alpha;\infty,1/w\right)D_j(1-s,-\alpha;1/w). \end{aligned} \quad (23.11)$$

24. We decompose the left side of (23.5) as

$$\begin{aligned} &\sum_{w_1|cd}\int_0^1\int_{y_0}^\infty \overline{\psi_j(\sigma_{1/w_1}(z))}E^*\left(\sigma_{1/w_1}(z),\tfrac12(1-\alpha)\right) \\ &\times E\left(\sigma_{1/w_1}(z),1/w;s-\tfrac12\alpha\right)d\mu(z) \\ &+ \int_{(\Gamma_0(cd)\backslash\mathcal{H})_{y_0}} \overline{\psi_j(z)}E^*\left(z,\tfrac12(1-\alpha)\right) \\ &\times E\left(z,1/w;s-\tfrac12\alpha\right)d\mu(z), \end{aligned} \quad (24.1)$$

where y_0 is chosen so that the remainder domain $(\Gamma_0(cd)\backslash\mathcal{H})_{y_0}$ is a compact set in $\mathcal{H}$. We then apply Lemma 7 and (23.7) to each term of (24.1). We obtain the crucial assertion

Lemma 10. *The functions*

$$\begin{aligned} &\left(1-\alpha^2\right)\left(s-\tfrac12\alpha\right)\left(1-s+\tfrac12\alpha\right)\Gamma(2s-\alpha) \\ &\times L(2s-\alpha,\chi_{cd})D_j(s,\alpha;1/w) \end{aligned} \quad (24.2)$$

of the complex variables s and α are all entire over $\mathbb{C}^2$.

In fact, it suffices to note that the multiple of (24.1) by the factor $\left(1-\alpha^2\right)\left(s-\frac12\alpha\right)\left(1-s+\frac12\alpha\right)\Gamma(2s-\alpha) \times L(2s-\alpha,\chi_{cd})$ is regular in s and α by Lemma 7.

On the other hand, we have, by (20.4),

$$\begin{aligned} &L(2s-\alpha,\chi_{cd})\varphi\left(s-\tfrac12\alpha;\infty,1/w\right) \\ &= \frac{1}{\pi}\varphi(w)\left(\frac{\pi}{cd}\right)^{2s-\alpha}\zeta(2(1-s)+\alpha) \\ &\times \frac{\Gamma\left(1-s+\frac12\alpha\right)}{\Gamma\left(s-\frac12\alpha\right)}\prod_{p|v}\left(p^{s-\frac12\alpha}-p^{1-s+\frac12\alpha}\right). \end{aligned} \quad (24.3)$$

Inserting this into (23.11), we get

$$\begin{aligned} &L(2s-\alpha,\chi_{cd})D_j(s,\alpha) \\ &= \frac{1}{\pi^2}\left(\frac{\pi^2}{cd}\right)^{2s-\alpha}\zeta(2(1-s)+\alpha)\frac{\Gamma(1-s,-\alpha;\kappa_j)}{\Gamma(s,\alpha;\kappa_j)} \\ &\times \sum_{w|cd}\varphi(w)\prod_{p|v}\left(p^{s-\frac12\alpha}-p^{1-s+\frac12\alpha}\right) \\ &\times D_j(1-s,-\alpha;1/w). \end{aligned} \quad (24.4)$$

We then let $\mathrm{Re}\,s$ be negative and so large that both $\zeta(2(1-s)+\alpha)$ and $D_j(1-s,-\alpha;1/w)$ are absolutely convergent. In this way we obtain, via Lemma 2, Stirling's formula, and the convexity argument,

Lemma 11. *Provided that* $\mathrm{Re}\,s$ *and* α *are bounded, we have*

$$\begin{aligned} &\left(1-\alpha^2\right)\left(s-\tfrac12\alpha\right)\left(1-s+\tfrac12\alpha\right) \\ &\times L(2s-\alpha,\chi_{cd})D_j(s,\alpha;1/w) \\ &\ll (\kappa_j+|s|+1)^\tau\exp\left(\tfrac12\pi\kappa_j\right), \end{aligned} \quad (24.5)$$

where τ depends only on $\mathrm{Re}\,s$ and $\mathrm{Re}\,\alpha$*, and the implied constant additionally on cd too.*

25. We still need to deal with the case $\epsilon_j = -1$. Here we shall have to overcome an additional technical difficulties, because Eisenstein series of non-zero weights naturally come up in our argument (see [11, Section 3.2]).

We introduce

$$\psi_j^-(z) = y(\partial_x - i\partial_y)\psi_j(z), \tag{25.1}$$

with our present vector ψ_j such that $\psi_j J = -\psi_j$. We have

$$\psi_j^-(\gamma(z)) = \psi_j^-(z)(\jmath(\gamma,z)/|\jmath(\gamma,z)|)^2, \quad \gamma\in\Gamma. \tag{25.2}$$

In fact, writing $\xi = \operatorname{Re}\lambda(z)$, $\eta = \operatorname{Im}\lambda(z)$ for a regular function λ, we have $(\partial_x - i\partial_y)[H(\lambda(z))] = \{(\partial H/\partial\xi) \times (\partial\xi/\partial x - i\partial\xi/\partial y) + (\partial H/\partial\eta)(\partial\eta/\partial x - i\partial\eta/\partial y)\} = \{(\partial H/\partial\xi - i\partial H/\partial\eta)(\partial\xi/\partial x - i\partial\xi/\partial y)\} = [(\partial_\xi - i\partial_\eta)H] \times (d\lambda/dz)$ by the Cauchy–Riemann equation applied to λ. We put $\lambda=\gamma$, $H=\psi_j$, and get $y(\partial_x - i\partial_y)\psi_j(z) = (y/\eta)(d\gamma/dz)\eta(\partial_\xi - i\partial_\eta)\psi_j(\xi+i\eta)$, which confirms (25.2).

To offset the automorphic factor in (25.2), we introduce

$$E_-(z,1/w;s) = \sum_{\gamma\in\Gamma_{1/w}\backslash\Gamma}\left(\operatorname{Im}\sigma_{1/w}^{-1}\gamma(z)\right)^s \times \left(\jmath(\sigma_{1/w}^{-1}\gamma,z)/|\jmath(\sigma_{1/w}^{-1}\gamma,z)|\right)^{-2}. \tag{25.3}$$

We should note the relation

$$y(\partial_x - i\partial_y)[E(z,1/w;s)] = -isE_-(z,1/w;s), \tag{25.4}$$

which can be confirmed by setting $\lambda = \sigma_{1/w}^{-1}\gamma$, $H = y^s$ in the above; and more precisely

$$\begin{aligned} &y(\partial_x - i\partial_y)[E(\sigma_{1/w_1}(z),1/w;s)] \\ &\quad = -isE_-(\sigma_{1/w_1}(z),1/w;s) \\ &\qquad \times \left(\jmath(\sigma_{1/w_1},z)/|\jmath(\sigma_{1/w_1},z)|\right)^{-2}. \end{aligned} \tag{25.5}$$

In particular, we have the functional equation

$$s\mathbb{E}_-(z,s) = (1-s)\mathbb{S}(s)\mathbb{E}_-(z,1-s), \tag{25.6}$$

with

$$\mathbb{E}_-(s) = \begin{pmatrix} \vdots \\ E_-(z,1/w;s) \\ \vdots \end{pmatrix}_{w|cd}. \tag{25.7}$$

Also, (25.5) implies that

$$\begin{aligned} &\Gamma(s+1)L(2s,\chi_{cd})E_-(\sigma_{1/w_2}(z),1/w_1;s) \\ &\qquad \ll y^{\operatorname{Re}s} + y^{1-\operatorname{Re}s}, \end{aligned} \tag{25.8}$$

as y tends to infinity while s remains bounded, which means that the left side is regular for all s, too. This is a counterpart of Lemma 7.

In the region of absolute convergence, we have, by (25.2),

$$\begin{aligned} &\int_{\Gamma\backslash\mathcal{H}} \overline{\psi_j^-(z)}E^*\left(z,\tfrac12(1-\alpha)\right)E_-\left(z,1/w;s-\tfrac12\alpha\right)d\mu(z) \\ &= \sum_{\gamma\in\Gamma_{1/w}\backslash\Gamma}\int_{\sigma_{1/w}^{-1}\gamma(\Gamma\backslash\mathcal{H})}\overline{\psi_j^-(\sigma_{1/w}(z))} \\ &\quad \times\left(\frac{\jmath(\gamma^{-1},\sigma_{1/w}(z))}{|\jmath(\gamma^{-1},\sigma_{1/w}(z))|}\right)^{-2}E^*\left(\sigma_{1/w}(z),\tfrac12(1-\alpha)\right) \\ &\quad \times y^{s-\frac12\alpha}\left(\frac{\jmath(\sigma_{1/w}^{-1}\gamma,\gamma^{-1}\sigma_{1/w}(z))}{|\jmath(\sigma_{1/w}^{-1}\gamma,\gamma^{-1}\sigma_{1/w}(z))|}\right)^{-2}d\mu(z) \\ &= \sum_{\gamma\in\Gamma_{1/w}\backslash\Gamma}\int_{\sigma_{1/w}^{-1}\gamma(\Gamma\backslash\mathcal{H})}\overline{\psi_j^-(\sigma_{1/w}(z))} \\ &\quad \times\left(\frac{\jmath(\sigma_{1/w},z)}{|\jmath(\sigma_{1/w},z)|}\right)^{2}E^*\left(\sigma_{1/w}(z),\tfrac12(1-\alpha)\right)y^{s-\frac12\alpha}d\mu(z) \\ &= \int_0^\infty\int_0^1\overline{(\partial_x - i\partial_y)[\psi_j(\sigma_{1/w}(z))]} \\ &\quad \times E^*\left(\sigma_{1/w}(z),\tfrac12(1-\alpha)\right)y^{s-\frac12\alpha-1}dxdy, \end{aligned} \tag{25.9}$$

since

$$\begin{aligned} \psi_j^-(\sigma_{1/w}(z)) &= y(\partial_x - i\partial_y)[\psi_j(\sigma_{1/w}(z))] \\ &\quad \times(\jmath(\sigma_{1/w},z)/|\jmath(\sigma_{1/w},z)|)^2. \end{aligned} \tag{25.10}$$

We observe then that $E^*\left(\sigma_{1/w}(z),\tfrac12(1-\alpha)\right)$ is even in x as (23.7) implies, and $\partial_y[\psi_j(\sigma_{1/w}(z))]$ is odd by (18.5). Hence (25.9) becomes

$$\begin{aligned} &\int_{\Gamma\backslash\mathcal{H}} \overline{\psi_j^-(z)}E^*\left(z,\tfrac12(1-\alpha)\right)E_-\left(z,1/w;s-\tfrac12\alpha\right)d\mu(z) \\ &\qquad = -i\frac{\Gamma(s+1,\alpha;\kappa_j)}{\pi^{s-\frac12\alpha}\Gamma\left(s+1-\frac12\alpha\right)}D_j(s,\alpha;1/w), \end{aligned} \tag{25.11}$$

provided absolute convergence holds throughout.

We decompose the left side of (25.11) in just the same way as we did in (24.1), and see, via (25.8), that

$$\begin{aligned} &(1-\alpha^2)\Gamma\left(s+1-\tfrac12\alpha\right)L(2s-\alpha,\chi_{cd}) \\ &\qquad \times D_j(s,\alpha;1/w) \end{aligned} \tag{25.12}$$

are all regular in both s and α. Also, (25.11) gives, via (25.6),

$$\begin{aligned} &\frac{\Gamma(s+1,\alpha;\kappa_j)}{\pi^{s-\frac12\alpha}\Gamma\left(s-\frac12\alpha\right)}\mathbb{D}_j(s,\alpha) \\ &= \frac{\Gamma(2-s,-\alpha;\kappa_j)}{\pi^{1-s+\frac12\alpha}\Gamma\left(1-s+\frac12\alpha\right)}\mathbb{S}\left(s-\tfrac12\alpha\right) \\ &\qquad \times\mathbb{D}_j(1-s,-\alpha), \end{aligned} \tag{25.13}$$

and in particular

$$D_j(s,\alpha) = \pi^{2s-\alpha-1}\frac{\Gamma(2-s,-\alpha;\kappa_j)}{\Gamma(s+1,\alpha;\kappa_j)}\frac{\Gamma\left(s-\frac12\alpha\right)}{\Gamma\left(1-s+\frac12\alpha\right)} \times\sum_{w|cd}\varphi\left(s-\tfrac12\alpha;\infty,1/w\right)D_j(1-s,-\alpha;1/w). \quad (25.14)$$

Hence, by (24.3), we have

$$L(2s-\alpha,\chi_{cd})D_j(s,\alpha) = \frac{1}{\pi^2}\left(\frac{\pi^2}{cd}\right)^{2s-\alpha}\zeta(2(1-s)+\alpha)\frac{\Gamma(2-s,-\alpha;\kappa_j)}{\Gamma(s+1,\alpha;\kappa_j)} \times\sum_{w|cd}\varphi(w)\prod_{p|v}\left(p^{s-\frac12\alpha}-p^{1-s+\frac12\alpha}\right) \times D_j(1-s,-\alpha;1/w). \quad (25.15)$$

With this, we obtain

Lemma 12. *With $\epsilon_j=-1$ as well, the assertions of Lemmas 10 and 11 hold.*

This ends our treatment of L_j and D_j. We omit the discussion of $L_{j,k}$, $D_{j,k}$, for they are analogous.

26. Now we may return to (21.3). Here we shall deal with the first term on the right, the contribution of real analytic cusp forms. Its contribution to $I(u,v,w,z;g;b/a)$ is, via (2.2), (2.3), (17.10), equal to

$$\frac{2}{a^v b^u (2\pi)^{u-w+1}}\sum_{c|a,d|b}c^{u+v}d^{\frac12(3u+v-w+z)} \times\sum_j R_j\left(\tfrac12(u+v+w+z-1),w+z-1\right) \times L_j\left(\tfrac12(u-v-w+z+1);1/c\right) \times\frac{([g]_++\epsilon_j[g]_-)(\kappa_j;u,v,w,z)}{\cosh\pi\kappa_j}. \quad (26.1)$$

with

$$R_j(s,\alpha)=\zeta(2s-\alpha)D_j(s,\alpha). \quad (26.2)$$

By Lemmas 9–12, we see readily that the expression (26.1) is meromorphic over $\mathbb{C}^4$, and especially in the vicinity of $p_{\frac12}$ it is regular; the necessary facts about $[g]_\pm$ is to be given shortly. Hence its value at $p_{\frac12}$ equals

$$\frac{1}{\pi\sqrt{ab}}\sum_{c|a,d|b}cd\sum_j R_j\left(\tfrac12,0\right)L_j\left(\tfrac12;1/c\right) \times\frac{([g]_++\epsilon_j[g]_-)(\kappa_j;p_{\frac12})}{\cosh\pi\kappa_j}. \quad (26.3)$$

We have another contribution of real analytic cusp forms that comes from J_-, which is, however, exactly the same as (26.3).

Let us make the last factor in (26.3) explicit. Thus, comparing (6.4) with [11, (4.3.13)–(4.3.14)], we see that the exchange of variables u and z is to be applied to [11, Sections 4.6–4.7] to get corresponding identities. More precisely, we have, under (3.4) and (4.1),

$$[g]_+(r;u,v,w,z)=\frac{1}{4\pi i}\cos(\tfrac12\pi(v-z)) \times\int_{(\eta_1)}\sin(\tfrac12\pi(u+v+w+z-2s)) \times\Gamma(\tfrac12(u+v+w+z-1)+ir-s) \times\Gamma(\tfrac12(u+v+w+z-1)-ir-s) \times\Gamma(s+1-w-z)\Gamma(s+1-v-w)g^*(s,w)ds, \quad (26.4)$$

$$[g]_-(r;u,v,w,z)=-\frac{1}{4\pi i}\cosh(\pi r) \times\int_{(\eta_1)}\cos(\pi(w+\tfrac12(v+z)-s)) \times\Gamma(\tfrac12(u+v+w+z-1)+ir-s) \times\Gamma(\tfrac12(u+v+w+z-1)-ir-s) \times\Gamma(s+1-w-z)\Gamma(s+1-v-w)g^*(s,w)ds, \quad (26.5)$$

corresponding to [11, (4.4.12)] and [ibid, (4.4.15)], respectively. We then put

$$\Phi_+(\xi;u,v,w,z;g)=-i(2\pi)^{w-v-2}\cos(\tfrac12\pi(v-z)) \times\int_{-i\infty}^{i\infty}\sin(\tfrac12\pi(u+v+w+z-2s)) \times\Gamma(\tfrac12(u+v+w+z-1)+\xi-s) \times\Gamma(\tfrac12(u+v+w+z-1)-\xi-s) \times\Gamma(s+1-w-z)\Gamma(s+1-v-w)g^*(s,w)ds; \quad (26.6)$$

$$\Phi_-(\xi;u,v,w,z;g)=i(2\pi)^{w-v-2}\cos(\pi\xi) \times\int_{-i\infty}^{i\infty}\cos(\pi(w+\tfrac12(v+z)-s)) \times\Gamma(\tfrac12(u+v+w+z-1)+\xi-s) \times\Gamma(\tfrac12(u+v+w+z-1)-\xi-s) \times\Gamma(s+1-w-z)\Gamma(s+1-v-w)g^*(s,w)ds; \quad (26.7)$$

and

$$\Xi(\xi;u,v,w,z;g) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\xi+\frac12(u+v+w+z-1)-s)}{\Gamma(\xi+\frac12(3-u-v-w-z)+s)} \times\Gamma(s+1-w-z)\Gamma(s+1-v-w)g^*(s,w)ds. \quad (26.8)$$

The paths in (26.6) and (26.7) are such that the poles of the first two gamma-factors and those of the other three factors in each integrand are separated to the right and the left, respectively, by the path, and ξ, u, v, w, z are assumed to be such that the path can be drawn. The path in (26.8) separates the poles of $\Gamma(\xi + \frac{1}{2}(u+v+w+z-1)-s)$ and those of $\Gamma(s+1-w-z)\Gamma(s+1-v-w)g^*(s,w)$ to the left and the right of the path, respectively. We have the relations

$$\Phi_+(\xi; u,v,w,z;g) = -\frac{(2\pi)^{w-u}\cos\left(\frac{1}{2}\pi(v-z)\right)}{4\sin(\pi\xi)} \times \{\Xi(\xi;u,v,w,z;g) - \Xi(-\xi;u,v,w,z;g)\}, \tag{26.9}$$

$$\Phi_-(\xi; u,v,w,z;g) = \frac{(2\pi)^{w-u}}{4\sin(\pi\xi)}\Big\{\sin(\pi(\tfrac{1}{2}(u-w)+\xi)) \times \Xi(\xi;u,v,w,z;g) - \sin(\pi(\tfrac{1}{2}(u-w)-\xi)) \times \Xi(-\xi;u,v,w,z;g)\Big\}, \tag{26.10}$$

provided the left sides are well-defined.

Under (4.1), we can obviously take (η_1) as the contours in the last three integrals; and we have, for $r\in\mathbb{R}$,

$$\begin{aligned}[g]_+(r;u,v,w,z) &= \frac{1}{2}(2\pi)^{1+u-w}\Phi_+(ir;u,v,w,z;g),\\ [g]_-(r;u,v,w,z) &= \frac{1}{2}(2\pi)^{1+u-w}\Phi_-(ir;u,v,w,z;g).\end{aligned} \tag{26.11}$$

In particular, we have, after continuation,

$$\begin{aligned}[g]_+(r;p_{\frac12}) &= -\frac{\pi}{4\sin(\pi i r)}\left(\Xi\left(ir;p_{\frac12};g\right) - \Xi\left(-ir;p_{\frac12};g\right)\right),\\ [g]_-(r;p_{\frac12}) &= \frac{\pi}{4}\left(\Xi\left(ir;p_{\frac12};g\right) + \Xi\left(-ir;p_{\frac12};g\right)\right),\end{aligned} \tag{26.12}$$

and

$$\left([g]_+ + \epsilon_j[g]_-\right)\left(r;p_{\frac12}\right) = \frac{\pi}{2}\mathrm{Re}\left\{\left(\epsilon_j + \frac{i}{\sinh \pi r}\right)\Xi\left(ir;p_{\frac12};g\right)\right\}, \tag{26.13}$$

since (3.2) and (26.8) imply $\overline{\Xi\left(ir;p_{\frac12};g\right)} = \Xi\left(-ir;p_{\frac12};g\right)$.

From this, we get immediately

Lemma 13. *Provided the polynomial A is supported by the set of square-free integers, the contribution of real analytic cusp forms to $M_2(g;A)$ is equal to*

$$\sum_{c,\,d}\mathcal{A}(c,d)\mathcal{C}(c,d;g), \tag{26.14}$$

where

$$\mathcal{A}(c,d) = \sum_{(ac,bd)=1}\frac{\alpha_{acl}\overline{\alpha_{bdl}}}{abl}, \tag{26.15}$$

and

$$\begin{aligned}\mathcal{C}(c,d;g) &= \sum_{\substack{j\\ \kappa_j^2+\frac14\in\mathrm{Sp}(\varGamma_0(cd))}}\frac{1}{\cosh\pi\kappa_j}R_j\left(\tfrac12,0\right)L_j\left(\tfrac12;1/c\right)\\ &\quad\times\mathrm{Re}\left\{\left(\epsilon_j+\frac{i}{\sinh\pi\kappa_j}\right)\Xi\left(i\kappa_j;p_{\frac12};g\right)\right\}.\end{aligned} \tag{26.16}$$

The fact that the parity symbol ϵ_j appears in this way will turn out to be crucial in our later discussion of a certain non-vanishing assertion (Sections 31–36).

The contribution of holomorphic cusp forms is analogous, and we may skip it.

27. We turn to the contribution of continuous spectrum; and we see from (21.3) that we need first to consider the sum

$$\begin{aligned}&\sum_n\frac{\sigma_{-2ir}(n;\chi_{cd})}{n^s}\times\prod_{p|(c_1,c)(d_1,d)}\left\{\sigma_{-2ir}(n_p)\left(1-\frac{1}{p^{1+2ir}}\right)-1\right\}\\ &=\sum_{l|(c_1,c)(d_1,d)}\mu((c_1,c)(d_1,d)/l)\prod_{p|l}\left(1-\frac{1}{p^{1+2ir}}\right)\\ &\quad\times\sum_n\sigma_{-2ir}(n;\chi_{cd})\sigma_{-2ir}(n_l)n^{-s},\end{aligned} \tag{27.1}$$

with $n_l = (n, l^\infty)$. We have

$$\begin{aligned}&\sum_n\sigma_{-2ir}(n;\chi_{cd})\sigma_{-2ir}(n_l)n^{-s}\\ &=\left\{\sum_{(n,l)=1}\frac{\sigma_{-2ir}(n;\chi_{cd})}{n^s}\right\}\left\{\sum_{n|l^\infty}\frac{\sigma_{-2ir}(n)}{n^s}\right\}\\ &=\zeta(s)L(s+2ir,\chi_{cd})\prod_{p|l}\left(1-\frac{1}{p^{s+2ir}}\right)^{-1}.\end{aligned} \tag{27.2}$$

Thus

$$\begin{aligned}&\sum_n\frac{\sigma_{-2ir}(n;\chi_{cd})}{n^s}\times\prod_{p|(c_1,c)(d_1,d)}\left\{\sigma_{-2ir}(n_p)\left(1-\frac{1}{p^{1+2ir}}\right)-1\right\}\\ &=\zeta(s)L(s+2ir,\chi_{cd})\sum_{l|(c_1,c)(d_1,d)}\mu((c_1,c)(d_1,d)/l)\\ &\quad\times\prod_{p|l}\left(1-\frac{1}{p^{s+2ir}}\right)^{-1}\left(1-\frac{1}{p^{1+2ir}}\right)\\ &=\zeta(s)L(s+2ir,\chi_{cd})\\ &\quad\times\prod_{p|(c_1,c)(d_1,d)}\left\{\left(1-\frac{1}{p^{s+2ir}}\right)^{-1}\times\left(1-\frac{1}{p^{1+2ir}}\right)-1\right\}.\end{aligned} \tag{27.3}$$

Next, we need to treat

$$\sum_n \frac{\sigma_{2ir}(n;\chi_{cd})\sigma_\alpha(n)}{n^s} \times \prod_{p|c_1}\left\{\sigma_{2ir}(n_p)\left(1-\frac{1}{p^{1-2ir}}\right)-1\right\}$$
$$=\sum_{l|c_1}\mu(c_1/l)\prod_{p|l}\left(1-\frac{1}{p^{1-2ir}}\right) \times \sum_n \frac{\sigma_{2ir}(n;\chi_{cd})\sigma_\alpha(n)\sigma_{2ir}(n_l)}{n^s}. \tag{27.4}$$

We have

$$\sum_n \frac{\sigma_{2ir}(n;\chi_{cd})\sigma_\alpha(n)\sigma_{2ir}(n_l)}{n^s} = \left\{\sum_{(n,l)=1}\frac{\sigma_{2ir}(n;\chi_{cd})\sigma_\alpha(n)}{n^s}\right\} \times \left\{\sum_{n|l^\infty}\frac{\sigma_{2ir}(n)\sigma_\alpha(n)}{n^s}\right\}. \tag{27.5}$$

Analogously to a famous formula of Ramanujan, we have

$$\sum_{(n,l)=1}\frac{\sigma_{2ir}(n;\chi_{cd})\sigma_\alpha(n)}{n^s} = \sum_{(n,cd)=1}\frac{\sigma_{2ir}(n)\sigma_\alpha(n)}{n^s}\sum_{n|(cd/l)^\infty}\frac{\sigma_\alpha(n)}{n^s}$$
$$=\frac{L(s,\chi_q)L(s-2ir,\chi_q)}{L(2s-2ir-\alpha,\chi_{cd})} \times L(s-\alpha,\chi_{cd})L(s-2ir-\alpha,\chi_{cd}) \times \prod_{p|cd/l}\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{1}{p^{s-\alpha}}\right)^{-1},$$

$$\sum_{n|l^\infty}\frac{\sigma_{2ir}(n)\sigma_\alpha(n)}{n^s} =\prod_{p|l}\frac{1-\frac{1}{p^{2s-2ir-\alpha}}}{\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{s-2ir}}\right)} \times \frac{1}{\left(1-\frac{1}{p^{s-\alpha}}\right)\left(1-\frac{1}{p^{s-2ir-\alpha}}\right)}. \tag{27.6}$$

Thus,

$$\sum_n \frac{\sigma_{2ir}(n;\chi_{cd})\sigma_\alpha(n)\sigma_{2ir}(n_l)}{n^s} =\frac{\zeta(s)L(s-2ir,\chi_{cd})\zeta(s-\alpha)L(s-2ir-\alpha,\chi_{cd})}{L(2s-2ir-\alpha,\chi_{cd})} \times\prod_{p|l}\frac{1-\frac{1}{p^{2s-2ir-\alpha}}}{\left(1-\frac{1}{p^{s-2ir}}\right)\left(1-\frac{1}{p^{s-2ir-\alpha}}\right)}. \tag{27.7}$$

Hence,

$$\sum_n \frac{\sigma_{2ir}(n;\chi_{cd})\sigma_\alpha(n)}{n^s} \times \prod_{p|c_1}\left\{\sigma_{2ir}(n_p)\left(1-\frac{1}{p^{1-2ir}}\right)-1\right\}$$
$$=\frac{\zeta(s)L(s-2ir,\chi_q)\zeta(s-\alpha)L(s-2ir-\alpha,\chi_{cd})}{L(2s-2ir-\alpha,\chi_{cd})} \times\prod_{p|c_1}\left\{\frac{\left(1-\frac{1}{p^{1-2ir}}\right)\left(1-\frac{1}{p^{2s-2ir-\alpha}}\right)}{\left(1-\frac{1}{p^{s-2ir}}\right)\left(1-\frac{1}{p^{s-2ir-\alpha}}\right)}-1\right\}$$
$$=\frac{\zeta(s)\zeta(s-2ir)\zeta(s-\alpha)\zeta(s-2ir-\alpha)}{\zeta(2s-2ir-\alpha)} \times\prod_{p|cd}\left(1-\frac{1}{p^{2s-2ir-\alpha}}\right)^{-1} \times\prod_{p|d_1}\left(1-\frac{1}{p^{s-2ir}}\right)\left(1-\frac{1}{p^{s-2ir-\alpha}}\right)$$
$$\times\prod_{p|c_1}\left\{\left(1-\frac{1}{p^{1-2ir}}\right)\left(1-\frac{1}{p^{2s-2ir-\alpha}}\right) -\left(1-\frac{1}{p^{s-2ir}}\right)\left(1-\frac{1}{p^{s-2ir-\alpha}}\right)\right\}. \tag{27.8}$$

28. Under the conditions (3.4), (4.1) and by (21.3), (27.3), (27.8), the contribution of the continuous spectrum to I via J_+^* is equal to

$$4\frac{(2\pi)^{w-u-2}}{a^v b^u}\int_{-\infty}^{\infty}\frac{Y_{a,b}(ir;u,v,w,z)Z(ir;u,v,w,z)}{\zeta(1+2ir)\zeta(1-2ir)} \times([g]_-+[g]_-)(r;u,v,w,z)dr \tag{28.1}$$

where

$$Z(\xi;u,v,w,z)=\zeta\left(\tfrac12(u+v+w+z-1)+\xi\right) \times\zeta\left(\tfrac12(u+v+w+z-1)-\xi\right) \times\zeta\left(\tfrac12(u+v-w-z+1)+\xi\right) \times\zeta\left(\tfrac12(u+v-w-z+1)-\xi\right) \times\zeta\left(\tfrac12(u-v-w+z+1)+\xi\right) \times\zeta\left(\tfrac12(u-v-w+z+1)-\xi\right) \tag{28.2}$$

and

$$Y_{a,b}(\xi;u,v,w,z)=\sum_{c|a,d|b}c^{u+v}d^{\frac12(3u+v-w+z-1)-\xi} \times X_{cd}(\xi;u,v,w,z), \tag{28.3}$$

with

$$X_{cd}(\xi;u,v,w,z)$$
$$=\prod_{p|cd}\left\{\left(1-\frac{1}{p^{1+2\xi}}\right)\left(1-\frac{1}{p^{1-2\xi}}\right)\left(1-\frac{1}{p^{u+v}}\right)\right\}^{-1}$$
$$\times\sum_{cd=c_1d_1}\frac{1}{d_1}\left(\frac{(d_1,d)}{(c_1,c)}\right)^{\frac12+\xi}$$
$$\times\prod_{p|(d_1,c)(c_1,d)}\left(1-\frac{1}{p^{\frac12(u-v-w+z+1)+\xi}}\right)$$
$$\times\prod_{p|(c_1,c)(d_1,d)}\left(\frac{1}{p^{\frac12(u-v-w+z)}}-\frac{1}{p^{\frac12+\xi}}\right)$$
$$\times\prod_{p|d_1}\left(1-\frac{1}{p^{\frac12(u+v+w+z-1)-\xi}}\right)$$
$$\cdot\left(1-\frac{1}{p^{\frac12(u+v-w-z+1)-\xi}}\right)$$
$$\times\prod_{p|c_1}\left\{\left(1-\frac{1}{p^{1-2\xi}}\right)\left(1-\frac{1}{p^{u+v}}\right)\right.$$
$$-\left(1-\frac{1}{p^{\frac12(u+v+w+z-1)-\xi}}\right)$$
$$\left.\cdot\left(1-\frac{1}{p^{\frac12(u+v-w-z+1)-\xi}}\right)\right\}. \tag{28.4}$$

One may carry out the last sum and transform X_{cd} and thus Y_{ab} into a more closed expression that is a product over prime divisors of ab; however, for our aim it does not seem particularly expedient to do so, and we leave (28.3) as it is.

To continue (28.1) to a neighborhood of $p_{\frac12}$, we need to shift the contour rightward and leftward appropriately as is done in [11, Section 4.7], and there appears a residual contribution, which will be treated in detail later. Here we shall compute, at $p_{\frac12}$, the integral thus continued.

By (28.4), we have, for $r\in\mathbb{R}$,

$$X_{cd}(ir;p_{\frac12})=\prod_{p|cd}\frac{1-\dfrac{1}{p^{\frac12+ir}}}{\left|1-\dfrac{1}{p^{1+2ir}}\right|^2\left(1-\dfrac1p\right)}$$
$$\times\sum_{cd=c_1d_1}\frac{1}{d_1}\left(\frac{(d_1,d)}{(c_1,c)}\right)^{\frac12+ir}\prod_{p|d_1}\left(1-\frac{1}{p^{\frac12-ir}}\right)^2$$
$$\times\prod_{p|c_1}\left\{\left(1-\frac{1}{p^{1-2ir}}\right)\left(1-\frac1p\right)\right.$$
$$\left.-\left(1-\frac{1}{p^{\frac12-ir}}\right)^2\right\}$$
$$=\prod_{p|cd}\frac{1-\dfrac{1}{p^{\frac12+ir}}}{\left|1-\dfrac{1}{p^{1+2ir}}\right|^2\left(1-\dfrac1p\right)}$$
$$\times\prod_{p|cd}\left\{\frac{(p,d)^{\frac12+ir}}{p}\left(1-\frac{1}{p^{\frac12-ir}}\right)^2\right.$$
$$+\frac{1}{(p,c)^{\frac12+ir}}\left(\left(1-\frac{1}{p^{1-2ir}}\right)\left(1-\frac1p\right)\right.$$
$$\left.\left.-\left(1-\frac{1}{p^{\frac12-ir}}\right)^2\right)\right\}$$
$$=c^{-\frac12-ir}\prod_{p|cd}\frac{1-\dfrac{1}{p^{\frac12+ir}}}{\left|1-\dfrac{1}{p^{1+2ir}}\right|^2\left(1-\dfrac1p\right)}$$
$$\times\left\{\left(1-\frac{1}{p^{1-2ir}}\right)\left(1-\frac1p\right)\right.$$
$$\left.-\left(1-\frac{1}{p^{\frac12-ir}}\right)^3\right\}$$
$$=c^{-\frac12-ir}\prod_{p|cd}\left|1+\frac{1}{p^{\frac12+ir}}\right|^{-2}\left(1-\frac1p\right)^{-1}$$
$$\times\left\{\left(1+\frac{1}{p^{\frac12-ir}}\right)\left(1-\frac1p\right)\right.$$
$$\left.-\left(1-\frac{1}{p^{\frac12-ir}}\right)^2\right\}. \tag{28.5}$$

This implies that

$$Y_{a,b}(ir;p_{\frac12})$$
$$=\sum_{c|a,d|b}(cd)^{\frac12-ir}\prod_{p|cd}\left|1+\frac{1}{p^{\frac12+ir}}\right|^{-2}\left(1-\frac1p\right)^{-1}$$
$$\times\left\{\left(1+\frac{1}{p^{\frac12-ir}}\right)\left(1-\frac1p\right)-\left(1-\frac{1}{p^{\frac12-ir}}\right)^2\right\}$$
$$=\frac{ab}{\varphi(ab)}\prod_{p|ab}\left(4\left|1+\frac{1}{p^{\frac12+ir}}\right|^{-2}-\frac1p\right). \tag{28.6}$$

We have obtained

Lemma 14. *Provided the polynomial A is supported by the set of square-free integers, the contribution of continuous spectrum to $M_2(g;A)$ is equal to*

$$\frac1\pi\sum_{(a,b)=1}\frac{\alpha_{al}\overline{\alpha_{bl}}}{\varphi(ab)l}\int_{-\infty}^{\infty}\frac{\left|\zeta\left(\frac12+ir\right)\right|^6}{|\zeta(1+2ir)|^2}$$
$$\times\prod_{p|ab}\left(4\left|1+\frac{1}{p^{\frac12+ir}}\right|^{-2}-\frac1p\right)$$
$$\times\operatorname{Re}\left\{\left(1+\frac{i}{\sinh\pi r}\right)\Xi\left(ir;p_{\frac12};g\right)\right\}dr. \tag{28.7}$$

29. We shall give the continuation procedure of (28.1) to a neighborhood of $p_{\frac{1}{2}}$. This is, however, analogous to that pertaining to the pure fourth moment $M_2(g;1)$ that is developed in [11, Sections 4.6–4.7]; and we can be brief.

By (26.9)–(26.11), we transform (28.1) into

$$\begin{aligned} &i\frac{(2\pi)^{w-u-1}}{a^v b^u}\int_{(0)}\frac{Z(\xi;u,v,w,z)}{\sin(\pi\xi)\zeta(1+2\xi)\zeta(1-2\xi)} \\ &\times\{Y_{a,b}(\xi;u,v,w,z)+Y_{a,b}(-\xi;u,v,w,z)\} \\ &\times\left\{\cos(\tfrac{1}{2}\pi(v-z))-\sin(\pi(\tfrac{1}{2}(u-w)+\xi))\right\} \\ &\times\Xi(\xi;u,v,w,z;g)d\xi; \end{aligned}\tag{29.1}$$

and applying the functional equation for ζ to $\zeta(1-2\xi)$, this becomes

$$\begin{aligned} &2i\frac{(2\pi)^{w-u-2}}{a^v b^u}\int_{(0)}\frac{(2\pi)^{2\xi}\Gamma(1-2\xi)Z(\xi;u,v,w,z)}{\zeta(2\xi)\zeta(1+2\xi)} \\ &\times\{Y_{a,b}(\xi;u,v,w,z)+Y_{a,b}(-\xi;u,v,w,z)\} \\ &\times\left\{\cos(\tfrac{1}{2}\pi(v-z))-\sin(\pi(\tfrac{1}{2}(u-w)+\xi))\right\} \\ &\times\Xi(\xi;u,v,w,z;g)d\xi \end{aligned}\tag{29.2}$$

(see [11, (4.6.14)–(4.6.15)]). We shift the last contour to the far right, and we obtain a meromorphic continuation to a domain containing the point $p_{\frac{1}{2}}$; then, restricting ourselves to the vicinity of $p_{\frac{1}{2}}$, we shift the contour back to the imaginary axis. The resulting integral has been considered already in the last section.

The residual contribution of the last procedure takes place when

$$\begin{aligned} \xi_1&=\frac{1}{2}(u+v+w+z-3), \\ \xi_2&=\frac{1}{2}(u-v-w+z-1), \\ \xi_3&=\frac{1}{2}(3-u-v-w-z), \\ \xi_4&=\frac{1}{2}(u+v-w-z-1). \end{aligned}\tag{29.3}$$

(see [11,(4.6.16)] and the bottom lines of [ibid, p. 173]). It should be stressed that this assertion depends on the fact that the singularities, save for those belonging to $Z(\xi;u,v,w,z)$, that we encounter in this procedure are independent of the location of (u,v,w,z); especially those of $Y_{a,b}(\pm\xi;u,v,w,z)$ come only from the first product on the right of (28.4) and are independent of (u,v,w,z).

REMARK 4. However, one should note that the set of poles of $Y_{a,b}(\xi;u,v,w,z)$ as a function of ξ cluster at the point $\xi=\pm\frac{1}{2}$ if a, b are allowed to vary arbitrarily. Thus, if the length of the polynomial A increases indefinitely, then the nature of the main term of $M_2(g;A)$ should become subtler.

30. With this, we have essentially finished spectrally decomposing $M_2(g;A)$. Although we have not yet computed the main term explicitly, the above is already quite adequate to analyze the error term in the asymptotic formula for the unweighted mean

$$\int_0^T\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4\left|A\left(\tfrac{1}{2}+it\right)\right|^2dt.\tag{30.1}$$

With this in mind, we shall investigate the location of poles of the Mellin transform $Z_2(s;A)$, focusing our attention to the contribution of real analytic cusp forms, for the relevant part of $Z_2(s;A)$ seems to be the most interesting.

Having the assertion of Lemma 13, the argument of [11, Section 5.3] works with $Z_2(s;A)$ as well without any essential change. We find, on the assumption on eigenvalues $\kappa_j^2+\frac{1}{4}$ made in the introduction, that

Lemma 15. *The function $Z_2(s;A)$ is meromorphic over the entire complex plane. It has a pole of the fifth order at $s=1$; and all other poles are in the half plane $\operatorname{Re}s\le\frac{1}{2}$. More precisely, $Z_2(s;A)$ has a pole at $\frac{1}{2}+i\kappa$, $\kappa>0$, if and only if it holds that*

$$\begin{aligned} &\sum_{c,d}\mathcal{A}(c,d) \\ &\times\sum R_j\left(\tfrac{1}{2},0\right)L_j\left(\tfrac{1}{2};1/c\right)\left(\epsilon_j-\frac{i}{\sinh\pi\kappa}\right)\ne0, \end{aligned}\tag{30.2}$$

in which the second sum is over $\kappa_j=\kappa$ with $\kappa_j^2+\frac{1}{4}\in\operatorname{Sp}(\varGamma_0(cd))$.

We are going to show that if A is fixed besides a natural condition on its coefficients, then (30.2) holds for infinitely many κ. To this end we shall establish in the sequel that there are infinitely many κ such that

$$\begin{aligned} \mathcal{R}(\kappa;A)&=\sum_{c,d}\mathcal{A}(c,d) \\ &\times\sum_{\substack{\kappa_j=\kappa\\ \kappa_j^2+\frac{1}{4}\in\operatorname{Sp}(\varGamma_0(cd))}}\epsilon_jR_j\left(\tfrac{1}{2},0\right)L_j\left(\tfrac{1}{2};1/c\right)\ne0. \end{aligned}\tag{30.3}$$

REMARK 5. As to the possible poles coming from the contribution of the continuous spectrum, one may follow the discussion in [11, p. 211]. In view of (28.7), we may have poles at

$$(2l+1)\frac{\pi i}{\log p},\quad l\in\mathbb{Z},\tag{30.4}$$

where $p|ab$ with $\alpha_a\alpha_b \neq 0$. Thus it can be asserted, somewhat informally, that as the length of A tends to infinity the imaginary axis is gradually filled up with poles of $Z_2(s;A)$.

31. To deal with $\mathcal{R}(\kappa;A)$, we adopt the argument of [11, Section 3.3]. Thus, on noting the definitions (17.1) and (26.2), we consider more generally the sum

$$\begin{aligned}&\mathcal{D}(u,v;h)\\&=\zeta(u+v)\sum_j \varrho_j(-f;1/c)D_j(u,u-v)\frac{h(\kappa_j)}{\cosh\pi\kappa_j}\\&=\zeta(u+v)\mathcal{D}_1(u,v;h),\end{aligned} \tag{31.1}$$

with an integer $f>0$, where the sum is extended over $\kappa_j^2+\frac14 \in \mathrm{Sp}(\Gamma_0(cd))$ with a fixed pair c,d, $\mu(cd)\neq 0$; also the weight h is assumed to be an even, entire function such that

$$h\left(\pm\tfrac12 i\right)=0 \tag{31.2}$$

and

$$h(r)\ll \exp(-c_0|r|^2), \tag{31.3}$$

with a certain $c_0>0$, in any fixed horizontal strip. By Lemmas 10–12, $\mathcal{D}(u,v;h)$ is meromorphic over $\mathbb{C}^2$, and regular in the vicinity of $\left(\frac12,\frac12\right)$; in particular, we have

$$\begin{aligned}&\mathcal{D}\left(\tfrac12,\tfrac12;h\right)\\&=\sum_j \epsilon_j\varrho_j(f;1/c)R_j\left(\tfrac12,0\right)\frac{h(\kappa_j)}{\cosh\pi\kappa_j}.\end{aligned} \tag{31.4}$$

In the region of absolute convergence, we have, by definition,

$$\begin{aligned}\mathcal{D}_1(u,v;h)&=\sum_m m^{-u}\sigma_{u-v}(m)\\&\times\sum_j \varrho_j(-f;1/c)\varrho_j(m;\infty)\frac{h(\kappa_j)}{\cosh\pi\kappa_j}.\end{aligned} \tag{31.5}$$

We apply (21.1) to the inner sum, getting

$$\mathcal{D}_1(u,v;h)=\mathcal{D}_2(u,v;h)+\mathcal{D}_3(u,v;h) \tag{31.6}$$

where

$$\begin{aligned}\mathcal{D}_2(u,v;h)&=\frac{1}{c\sqrt{d}}\sum_m m^{-u}\sigma_{u-v}(m)\\&\times\sum_{\substack{l\\(l,d)=1}}\frac1l S(m,-\bar{d}f;cl)\psi\left(\frac{4\pi}{cl\sqrt{d}}\sqrt{mf}\right),\end{aligned} \tag{31.7}$$

with

$$\psi(x)=\frac{4}{\pi^2}\int_{-\infty}^{\infty} r\sinh(\pi r)K_{2ir}(x)h(r)dr, \tag{31.8}$$

and

$$\begin{aligned}&L(u+v,\chi_{cd})\mathcal{D}_3(u,v;h)\\&=-\frac1\pi\sum_{c_1d_1=cd}\frac{1}{d_1}\int_{-\infty}^{\infty}\left(\frac{(d_1,d)}{\sqrt{d}}\right)^{1+2ir}\frac{f^{ir}\sigma_{-2ir}(f;\chi_{cd})}{|L(1+2ir,\chi_{cd})|^2}\\&\times\zeta(u+ir)\zeta(u-ir)\zeta(v+ir)\zeta(v-ir)\\&\times\prod_{p|(c_1,c)(d_1,d)}\left(\sigma_{-2ir}(f_p)\left(1-\frac{1}{p^{1+2ir}}\right)-1\right)\\&\times\prod_{p|d_1}\left(1-\frac{1}{p^{u-ir}}\right)\left(1-\frac{1}{p^{v-ir}}\right)\\&\times\prod_{p|c_1}\left\{\left(1-\frac{1}{p^{1-2ir}}\right)\left(1-\frac{1}{p^{u+v}}\right)\right.\\&\qquad\left.-\left(1-\frac{1}{p^{u-ir}}\right)\left(1-\frac{1}{p^{v-ir}}\right)\right\}h(r)dr,\end{aligned} \tag{31.9}$$

in which we have used (27.8) with $s=u+ir$, $\alpha=u-v$.

32. To transform $\mathcal{D}_2$ we use the formula

$$\psi(x)=\frac{1}{\pi^2}\int_{(\alpha)}\frac{\hat{h}(s)}{\cos\pi s}\left(\frac{x}{2}\right)^{-2s}ds,\qquad 0<\alpha<\tfrac12, \tag{32.1}$$

where

$$\hat{h}(s)=\int_{-\infty}^{\infty} rh(r)\frac{\Gamma(s+ir)}{\Gamma(1-s+ir)}dr \tag{32.2}$$

(see [11, p. 113]). Moving the last path far down, we see that $\hat{h}$ is entire. Also we have

$$\hat{h}(\pm\tfrac12)=0, \tag{32.3}$$

and (32.1) is replaced by

$$\psi(x)=\frac{1}{\pi^2}\int_{(\alpha)}\frac{\hat{h}(s)}{\cos\pi s}\left(\frac{x}{2}\right)^{-2s}ds, \tag{32.4}$$

where $-\frac32<\alpha<\frac32$. The integrand decays exponentially, which facilitate our discussion greatly. We stress that the presence of the factor ϵ_j in (30.3) has induced this effect.

Thus in (31.7) we have

$$\begin{aligned}&\sum_{\substack{l\\(l,d)=1}}\frac1l S(m,-\bar{d}f;cl)\psi\left(\frac{4\pi}{cl\sqrt{d}}\sqrt{mf}\right),\\&=\frac{1}{\pi^2}\sum_{\substack{l\\(l,d)=1}}\frac1l S(m,-\bar{d}f;cl)\\&\times\int_{(\alpha)}\frac{\hat{h}(s)}{\cos\pi s}\left(\frac{2\pi}{cl\sqrt{d}}\sqrt{mf}\right)^{-2s}ds,\end{aligned} \tag{32.5}$$

with

$$-\tfrac{3}{2} < \alpha < -\tfrac{1}{4}. \tag{32.6}$$

The right side of (32.5) converges absolutely. Then we assume that

$$\operatorname{Re} u,\ \operatorname{Re} v > 1 - \alpha. \tag{32.7}$$

On this we insert (32.5) into (31.7), and get

$$\mathcal{D}_2(u,v;h) = \frac{1}{\pi^2 c\sqrt{d}} \sum_{\substack{l \\ (l,d)=1}} \frac{1}{l} P(u,v;l), \tag{32.8}$$

where

$$\begin{aligned} P(u,v;l) &= \int_{(\alpha)} \Big(\frac{2\pi}{cl\sqrt{d}}\sqrt{f}\Big)^{-2s} \frac{\hat{h}(s)}{\cos(\pi s)} \\ &\times \sum_{\substack{a=1 \\ (a,cl)=1}}^{cl} \exp(-2\pi i \bar{d} f a/cl) \\ &\times \sum_{m=1}^{\infty} \sigma_{u-v}(m) \exp(2\pi i m \bar{a}/cl) m^{-u-s} ds, \end{aligned} \tag{32.9}$$

with $a\bar{a} \equiv 1 \bmod cl$.

We introduce further a sub-region of (32.7):

$$\begin{aligned} &1 - \alpha < \operatorname{Re}(u), \operatorname{Re}(v) < -\beta, \\ &-\tfrac{3}{2} < \beta < \alpha - 1, \quad -\tfrac{1}{2} < \alpha < -\tfrac{1}{4}. \end{aligned} \tag{32.10}$$

Then we move the path in (32.9) to (β). On the assumption $u \neq v$, we have, by Estermann's functional equation (see [11, Lemma 3.7]),

$$\begin{aligned} P(u,v;l) &= -2\pi i c_{cl}(f)(cl)^{1-u-v} \\ &\times \Big\{(2\pi\sqrt{f/d})^{2(u-1)}\hat{h}(1-u)\zeta(1-u+v)/\cos\pi u \\ &+ (2\pi\sqrt{f/d})^{2(v-1)}\hat{h}(1-v)\zeta(1-v+u)/\cos\pi v\Big\} \\ &+ 2(2\pi)^{u+v-2}(cl)^{1-u-v} \\ &\times \Big\{ \sum_{m=1}^{\infty} m^{u-1}\sigma_{v-u}(m) c_{cl}(dm+f)\Psi_+(u,v;dm/f;h) \\ &+ \sum_{m=1}^{\infty} m^{u-1}\sigma_{v-u}(m) c_{cl}(dm-f) \\ &\qquad \times \Psi_-(u,v;dm/f;h)\Big\}, \end{aligned} \tag{32.11}$$

where

$$\begin{aligned} \Psi_+(u,v;x;h) &= -\int_{(\beta)} \Gamma(1-u-s)\Gamma(1-v-s) \\ &\qquad \times \cos\left(\pi\left(s + \tfrac{1}{2}(u+v)\right)\right) \frac{\hat{h}(s)}{\cos\pi s} x^s ds \end{aligned} \tag{32.12}$$

and

$$\begin{aligned} \Psi_-(u,v;x;h) &= \cos\left(\tfrac{1}{2}\pi(u-v)\right) \\ &\times \int_{(\beta)} \Gamma(1-u-s)\Gamma(1-v-s) \frac{\hat{h}(s)}{\cos(\pi s)} x^s ds. \end{aligned} \tag{32.13}$$

33. We insert (32.11) into (32.8). We get under (32.10) that

$$\begin{aligned} &L(u+v,\chi_{cd})\mathcal{D}_2(u,v;h) \\ &= \left\{\mathcal{D}_2^1 + \mathcal{D}_2^2 + \mathcal{D}_2^3 + \mathcal{D}_2^4\right\}(u,v;h), \end{aligned} \tag{33.1}$$

where

$$\begin{aligned} \mathcal{D}_2^1 &= \frac{2}{\pi i\sqrt{d}}\Bigg\{ \left(2\pi\sqrt{\frac{f}{d}}\right)^{2(u-1)} \frac{\hat{h}(1-u)}{\cos\pi u}\zeta(1-u+v) \\ &+ \left(2\pi\sqrt{\frac{f}{d}}\right)^{2(v-1)} \frac{\hat{h}(1-v)}{\cos\pi v}\zeta(1-v+u)\Bigg\} \\ &\times \sigma_{1-u-v}(f,\chi_{cd}) \prod_{p|c}\left(\sigma_{1-u-v}(f_p)\left(1 - \frac{1}{p^{u+v}}\right) - 1\right), \\ \mathcal{D}_2^2 &= 8\frac{(2\pi)^{u+v-4}}{\sqrt{d}} \sum_m m^{u-1}\sigma_{v-u}(m) \\ &\times \sigma_{1-u-v}(dm+f;\chi_{cd}) Psi_+(u,v;dm/f;h) \\ &\times \prod_{p|c}\left(\sigma_{1-u-v}((dm+f)_p)\left(1 - \frac{1}{p^{u+v}}\right) - 1\right), \\ \mathcal{D}_2^3 &= 8\frac{(2\pi)^{u+v-4}}{\sqrt{d}} \sum_{\substack{m \\ dm\neq f}} m^{u-1}\sigma_{v-u}(m) \\ &\times \sigma_{1-u-v}(dm-f;\chi_{cd})\Psi_-(u,v;dm/f;h) \\ &\times \prod_{p|c}\left(\sigma_{1-u-v}((dm-f)_p)\left(1 - \frac{1}{p^{u+v}}\right) - 1\right), \\ \mathcal{D}_2^4 &= 8(2\pi)^{u+v-4}\frac{\varphi(c)}{c^{u+v}\sqrt{d}} L(u+v-1,\chi_d)(f/d)^{u-1} \\ &\times \sigma_{v-u}(f/d)\Psi_-(u,v;1;h), \end{aligned} \tag{33.2}$$

in which $\mathcal{D}_2^4$ appears only when $d|f$.

The expansion (33.1) with (33.2) has been proved under the assumption that $u \neq v$ and (32.10) holds. However, the former can be dropped now; and also $\mathcal{D}_2^2$ and $\mathcal{D}_2^3$ converge absolutely if $1 + \beta < \operatorname{Re} u,\ \operatorname{Re} v < -\beta$. In particular, $L(u+v,\chi_{cd})\mathcal{D}_2(u,v;h)$ is regular at $\left(\frac{1}{2},\frac{1}{2}\right)$, and there (33.1) holds.

Further, shifting the path in (31.9) upward and downward appropriately, we have the following continuation of $\mathcal{D}_3$ to the domain $\operatorname{Re} u,\ \operatorname{Re} v < 1$:

$$\begin{aligned} &L(u+v,\chi_{cd})\mathcal{D}_3(u,v;h) \\ &= \left\{\mathcal{D}_3^1 + \mathcal{D}_3^2 + \mathcal{D}_3^3\right\}(u,v;h), \end{aligned} \tag{33.3}$$

where

$$\mathcal{D}_3^1 = -\frac{1}{\pi}\sum_{c_1d_1=cd}\frac{1}{d_1}\int_{-\infty}^{\infty}\left(\frac{(d_1,d)}{\sqrt{d}}\right)^{1+2ir}$$
$$\times \frac{f^{ir}\sigma_{-2ir}(f;\chi_{cd})}{|L(1+2ir,\chi_{cd})|^2}$$
$$\times \zeta(u+ir)\zeta(u-ir)\zeta(v+ir)\zeta(v-ir)$$
$$\times \prod_{p|(c_1,c)(d_1,d)}\left(\sigma_{-2ir}(f_p)\left(1-\frac{1}{p^{1+2ir}}\right)-1\right)$$
$$\times \prod_{p|d_1}\left(1-\frac{1}{p^{u-ir}}\right)\left(1-\frac{1}{p^{v-ir}}\right)$$
$$\times \prod_{p|c_1}\left\{\left(1-\frac{1}{p^{1-2ir}}\right)\left(1-\frac{1}{p^{u+v}}\right)\right.$$
$$\left. -\left(1-\frac{1}{p^{u-ir}}\right)\left(1-\frac{1}{p^{v-ir}}\right)\right\}h(r)dr,$$
$$\mathcal{D}_3^2 = -2f^{1-u}\sigma_{2(u-1)}(f;\chi_{cd})$$
$$\times \frac{\zeta(u+v-1)\zeta(v-u+1)}{L(3-2u,\chi_{cd})}h(i(u-1))$$
$$\times \sum_{c_1d_1=cd}\frac{\varphi(c_1)}{c_1^{u+v}d_1}\left(\frac{(d_1,d)}{\sqrt{d}}\right)^{3-2u}$$
$$\times \prod_{p|(c_1,c)(d_1,d)}\left(\sigma_{2(u-1)}(f_p)\left(1-\frac{1}{p^{3-2u}}\right)-1\right)$$
$$\times \prod_{p|d_1}\left(1-\frac{1}{p^{u+v-1}}\right)$$
$$-2f^{1-v}\sigma_{2(v-1)}(f;\chi_{cd})$$
$$\times \frac{\zeta(u+v-1)\zeta(u-v+1)}{L(3-2v,\chi_{cd})}h(i(v-1))$$
$$\times \sum_{c_1d_1=cd}\frac{\varphi(c_1)}{c_1^{u+v}d_1}\left(\frac{(d_1,d)}{\sqrt{d}}\right)^{3-2v}$$
$$\times \prod_{p|(c_1,c)(d_1,d)}\left(\sigma_{2(v-1)}(f_p)\left(1-\frac{1}{p^{3-2v}}\right)-1\right)$$
$$\times \prod_{p|d_1}\left(1-\frac{1}{p^{u+v-1}}\right),$$
$$\mathcal{D}_3^3 = -2f^{u-1}\sigma_{2(1-u)}(f;\chi_{cd})$$
$$\times \frac{\zeta(u+v-1)\zeta(v-u+1)}{L(3-2u,\chi_{cd})}h(i(u-1))$$
$$\times \prod_{p|cd}\left(1-\frac{1}{p^{2u-1}}\right)^{-1}$$
$$\times \sum_{c_1d_1=cd}\frac{\varphi(d_1)}{d_1^2}\left(\frac{(d_1,d)}{\sqrt{d}}\right)^{2u-1}$$
$$\times \prod_{p|(c_1,c)(d_1,d)}\left(\sigma_{2(1-u)}(f_p)\left(1-\frac{1}{p^{2u-1}}\right)-1\right)$$
$$\times \prod_{p|d_1}\left(1-\frac{1}{p^{v-u+1}}\right)$$
$$\times \prod_{p|c_1}\left\{\left(1-\frac{1}{p^{3-2u}}\right)\left(1-\frac{1}{p^{u+v}}\right)\right.$$
$$\left. -\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{v-u+1}}\right)\right\},$$
$$-2f^{v-1}\sigma_{2(1-v)}(f;\chi_{cd})\frac{\zeta(u+v-1)\zeta(u-v+1)}{L(3-2v,\chi_{cd})}$$
$$\times h(i(v-1))\prod_{p|cd}\left(1-\frac{1}{p^{2v-1}}\right)^{-1}$$
$$\times \sum_{c_1d_1=cd}\frac{\varphi(d_1)}{d_1^2}\left(\frac{(d_1,d)}{\sqrt{d}}\right)^{2v-1}$$
$$\times \prod_{p|(c_1,c)(d_1,d)}\left(\sigma_{2(1-v)}(f_p)\left(1-\frac{1}{p^{2v-1}}\right)-1\right)$$
$$\times \prod_{p|d_1}\left(1-\frac{1}{p^{u-v+1}}\right)$$
$$\times \prod_{p|c_1}\left\{\left(1-\frac{1}{p^{3-2v}}\right)\left(1-\frac{1}{p^{u+v}}\right)\right.$$
$$\left. -\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{u-v+1}}\right)\right\}. \tag{33.4}$$

We see readily that $\mathcal{D}_3^1$ and $\mathcal{D}_3^2$ are regular at $\left(\frac{1}{2},\frac{1}{2}\right)$. As to $\mathcal{D}_3^3$, the factors $\prod_{p|cd}(1-p^{1-2u})^{-1}$ and $\prod_{p|cd}(1-p^{1-2v})^{-1}$ diverge at the point unless $cd=1$; however, $\mathcal{D}_3^3$ itself must be regular there, for $L(u+v,\chi_{cd})\mathcal{D}_1$, $L(u+v,\chi_{cd})\mathcal{D}_2$ are regular, and thus $L(u+v,\chi_{cd})\mathcal{D}_3$ as well.

Hence, from (31.1), (31.6), (33.1), and (33.3), we obtain

$$\mathcal{D}\left(\tfrac{1}{2},\tfrac{1}{2};h\right) = \frac{cd}{\varphi(cd)}\Big\{\mathcal{D}_2^1+\mathcal{D}_2^2+\mathcal{D}_2^3$$
$$+\mathcal{D}_2^4+\mathcal{D}_3^1+\mathcal{D}_3^2+\mathcal{D}_3^3\Big\}\left(\tfrac{1}{2},\tfrac{1}{2};h\right). \tag{33.5}$$

34. The last equation gives

Lemma 16. *We have, with the weight h as above,*

$$\sum_j \epsilon_j\varrho_j(f;1/c)R_j\left(\tfrac{1}{2},0\right)\frac{h(\kappa_j)}{\cosh\pi\kappa_j}$$
$$=\sum_{a=1}^{7}\mathcal{H}_a(f;h), \tag{34.1}$$

where

$$\mathcal{H}_1 = \frac{2cd}{\pi^3 i\varphi(cd)}$$
$$\times\left\{(c_E-\log(2\pi\sqrt{f/d}))(\hat{h})'(\tfrac{1}{2})+\tfrac{1}{4}(\hat{h})''(\tfrac{1}{2})\right\}$$

$$\times\, \tau(f,\chi_{cd}) f^{-\frac12} \prod_{p|c} \left(\tau(f_p)\left(1-\frac{1}{p}\right)-1\right),$$

$$\mathcal{H}_2 = \frac{c\sqrt{d}}{\pi^3\varphi(cd)} \times \sum_m m^{-\frac12}\tau(m)\tau(dm+f;\chi_{cd})\Psi_+(dm/f;h) \times \prod_{p|c}\left(\tau((dm+f)_p)\left(1-\frac{1}{p}\right)-1\right),$$

$$\mathcal{H}_3 = \frac{c\sqrt{d}}{\pi^3\varphi(cd)} \times \sum_{\substack{m\\ dm\neq f}} m^{-\frac12}\tau(m)\tau(dm-f;\chi_{cd})\Psi_-(dm/f;h) \times \prod_{p|c}\left(\tau((dm-f)_p)\left(1-\frac{1}{p}\right)-1\right),$$

$$\mathcal{H}_4 = -\frac{\delta_{d,1}}{2\pi^3} f^{-\frac12}\tau(f)\Psi_-(1;h),$$

$$\mathcal{H}_a = \frac{cd}{\varphi(cd)}\mathcal{D}_3^{a-4}\left(\tfrac12,\tfrac12;h\right), \quad 5\le a\le 7. \tag{34.2}$$

Here τ is the divisor function, $\Psi_\pm(x;h) = \Psi_\pm\left(\frac12,\frac12;x;h\right)$, and $\mathcal{H}_4$ vanishes unless $d=1$.

This is a counterpart of [11, Lemma 3.8], and follows immediately from (31.4), (33.2) and (33.5). We have left $\mathcal{H}_a(f;h)$, $5\le a\le 7$, without computing it explicitly, because it seems better to avoid the highly complicated computation of $\mathcal{D}_3^3\left(\frac12,\frac12;h\right)$ caused by the two products over $p|cd$ mentioned above; and in fact those $\mathcal{H}_a(f;h)$ will readily turn out to be negligible in our application of (34.1) to be given in the next section.

From [11, pp. 119–121], we quote the following:

$$\begin{aligned}(\hat h)'(\tfrac12) &= 2\int_{-\infty}^{\infty} rh(r)\frac{\Gamma'}{\Gamma}(\tfrac12+ir)dr,\\ (\hat h)''(\tfrac12) &= 4\int_{-\infty}^{\infty} rh(r)\Big\{\frac{\Gamma'}{\Gamma}(\tfrac12+ir)\Big\}^2 dr,\end{aligned} \tag{34.3}$$

$$\Psi_+(x;h) = 2\pi\int_0^1 \{y(1-y)(1+y/x)\}^{-\frac12} \times \int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\Big\{\frac{y(1-y)}{x+y}\Big\}^{ir} dr dy. \tag{34.4}$$

For $x>1$

$$\Psi_-(x;h) = 2\pi i\int_0^1 \{y(1-y)(1-y/x)\}^{-\frac12} \times \int_{-\infty}^{\infty} \frac{rh(r)}{\cosh(\pi r)}\Big\{\frac{y(1-y)}{x-y}\Big\}^{ir} dr dy. \tag{34.5}$$

For $x=1$

$$\Psi_-(1;h) = 2\pi^2\int_{-\infty}^{\infty} rh(r)\frac{\sinh(\pi r)}{(\cosh(\pi r))^2}dr. \tag{34.6}$$

For $0<x<1$

$$\Psi_-(x;h) = \int_0^\infty \Big\{\int_{(\beta)} x^s(y(y+1))^{s-1}\frac{\Gamma(\frac12-s)^2}{\Gamma(1-2s)\cos(\pi s)}ds\Big\} \times \Big\{\int_{-\infty}^{\infty} rh(r)\Big(\frac{y}{1+y}\Big)^{ir}dr\Big\}dy, \tag{34.7}$$

where $-\frac32<\beta<\frac12$, $\beta\neq-\frac12$.

35. We shall continue our discussion, adopting the argument given in [11, pp. 124–130]. Thus we first state the following approximation for $L_j\left(\frac12;1/c\right)$: Let K tend to infinity, and assume that

$$|\kappa_j - K| \le G\log K \tag{35.1}$$

with

$$K^{\frac12+\delta} < G < K^{1-\delta}, \quad 0<\delta<\tfrac12. \tag{35.2}$$

Then we have, for any $N\ge1$ and $\lambda = C\log K$ with a sufficiently large $C>0$,

$$\begin{aligned} L_j\left(\tfrac12;1/c\right) &= \sum_{f\le 3K\sqrt{cd}} \varrho_j(f;1/c)f^{-\frac12}\exp(-(f/(K\sqrt{cd}))^\lambda)\\ &\quad -\sum_{\nu=0}^{N_1}\sum_{f\le 3K\sqrt{cd}} \varrho_j(f;1/c)f^{-\frac12}U_\nu(f/(K\sqrt{cd}))\\ &\quad\times(1-(\kappa_j/K)^2)^\nu + O(K^{-\frac15 N}+K^{-\frac12 C}),\end{aligned} \tag{35.3}$$

with the implied constant depending only on δ, C, and N. Here $N_1=[3N/\delta]$ and

$$U_\nu(x) = \frac{1}{2\pi i\lambda}\int_{(-\lambda^{-1})}(4\pi^2x)^w u_\nu(w)\Gamma(w/\lambda)dw, \tag{35.4}$$

where $u_\nu(w)$ is a polynomial of degree $\le 2N_1$, whose coefficients are independent of κ_j and bounded by a constant depending only on δ and N.

In fact, this assertion is a counterpart of [11, Lemma 3.9] and the proof is analogous; the necessary change is only in that we now use (23.13) instead under the assumption $\varpi_j\epsilon_j=+1$ as $L\left(\frac12;1/c\right)=0$ if $\varpi_j\epsilon_j=-1$.

With this, we now set, in (34.1),

$$h(r) = \left(r^2+\tfrac14\right)\Big\{\exp(-((r-K)/G)^2) + \exp(-((r+K)/G)^2)\Big\}. \tag{35.5}$$

We have

$$(\hat{h})'\left(\tfrac{1}{2}\right) = 2i\pi^{\frac{3}{2}}K^3G + O(KG^3),$$
$$(\hat{h})''\left(\tfrac{1}{2}\right) = 8i\pi^{\frac{3}{2}}K^3G\log K + O(KG^3\log K). \quad (35.6)$$

(see [11, p. 129]).

We have, by (34.1) and (35.4),

$$\sum_j \epsilon_j R_j\left(\tfrac{1}{2},0\right)L_j\left(\tfrac{1}{2};1/c\right)\frac{h(\kappa_j)}{\cosh\pi\kappa_j}$$
$$= \sum_{f\le 3K\sqrt{cd}} f^{-\frac{1}{2}}\exp(-(f/(K\sqrt{cd})^{\lambda})\sum_{a=1}^{7}\mathcal{H}_a(f;h)$$
$$-\sum_{\nu\le N_1}\sum_{f\le 3K\sqrt{cd}} f^{-\frac{1}{2}}U_\nu(f/(K\sqrt{cd}))$$
$$\times\sum_{a=1}^{7}\mathcal{H}_a(f;h_\nu) + O(1), \quad (35.7)$$

where the five terms correspond to those on the right side of (34.1), respectively, with the present h and $h_\nu(r) = h(r)(1-(r/K)^2)^\nu$. Since we have imposed (35.1)–(35.2), those terms with $\nu\ge 1$ can actually be ignored, and it suffices to consider instead

$$\sum_{f\le 3K\sqrt{cd}} f^{-\frac{1}{2}}\exp(-(f/(K\sqrt{cd})^{\lambda})\sum_{a=1}^{7}\mathcal{H}_a(f;h)$$
$$-\sum_{f\le 3K\sqrt{cd}} f^{-\frac{1}{2}}U_0(f/(K\sqrt{cd}))\sum_{a=1}^{7}\mathcal{H}_a(f;h). \quad (35.8)$$

The discussion in [11, pp. 128–129] works just fine with our present situation as well; and the contribution of $\mathcal{H}_a$, $a = 2,3,4$, turns out to be negligible.

REMARK 6. However, if the uniformity in the Stufe cd is required, then this part of our argument should become subtle.

As to $\mathcal{H}_1$, its contribution to (35.8) is equal to

$$\frac{4cd}{\pi^{\frac{3}{2}}\varphi(cd)}K^3G\,(\mathcal{K}_1+\mathcal{K}_2) + O(KG^3(\log K)^2), \quad (35.9)$$

where we have used (35.6), and

$$\mathcal{K}_1 = \sum_f\left(c_E - \log(2\pi\sqrt{f/d}) + \log K\right)$$
$$\times\exp(-(f/(K\sqrt{cd})^{\lambda})$$
$$\times\frac{\tau(f;\chi_{cd})}{f}\prod_{p|c}\left(\tau(f_p)\left(1-\frac{1}{p}\right)-1\right),$$
$$\mathcal{K}_2 = -\sum_f\left(c_E - \log(2\pi\sqrt{f/d}) + \log K\right)$$
$$\times U_0(f/(K\sqrt{cd}))$$
$$\times\frac{\tau(f;\chi_{cd})}{f}\prod_{p|c}\left(\tau(f_p)\left(1-\frac{1}{p}\right)-1\right). \quad (35.10)$$

To compute $\mathcal{K}_1, \mathcal{K}_2$, let us put

$$z(s) = \sum_f\frac{\tau(f;\chi_{cd})}{f^{s+1}}$$
$$\times\prod_{p|c}\left(\tau(f_p)\left(1-\frac{1}{p}\right)-1\right). \quad (35.11)$$

Then

$$\mathcal{K}_1$$
$$= \frac{1}{2\pi i\lambda}\int_{(1)}\left\{(\log(K\sqrt{d}/2\pi)+c_E)z(s)+\tfrac{1}{2}z'(s)\right\}$$
$$\times(K\sqrt{cd})^s\Gamma(s/\lambda)ds,$$
$$\mathcal{K}_2$$
$$= -\frac{1}{2\pi i\lambda}\int_{(-1)}\left\{(\log(K\sqrt{d}/2\pi)+c_E)z(-s)+\tfrac{1}{2}z'(-s)\right\}$$
$$\times(4\pi^2/K\sqrt{cd})^s u_0(w)\Gamma(s/\lambda)ds. \quad (35.12)$$

The latter can be replace by

$$-\frac{1}{2\pi i\lambda}\int_{(-1)}\left\{(\log(K\sqrt{d}/2\pi)+c_E)z(-s)+\tfrac{1}{2}z'(-s)\right\}$$
$$\times(4\pi^2/K\sqrt{cd})^s\Gamma(s/\lambda)ds \quad (35.13)$$

with an admissible error (see [11, p. 127] for a description of u_0).

We have

$$z(s) = \zeta(s+1)^2\prod_{p|d}\left(1-\frac{1}{p^{s+1}}\right)$$
$$\times\prod_{p|c}\left(1-\frac{1}{p^{s+1}}-\frac{2}{p^{s+2}}+\frac{1}{p^{2s+2}}+\frac{1}{p^{2s+3}}\right). \quad (35.14)$$

Hence, we get

$$\mathcal{K}_1,\ \mathcal{K}_2\sim\frac{1}{3}(\log K)^3\frac{\varphi(cd)}{cd}\prod_{p|c}\left(1-\frac{1}{p^2}\right). \quad (35.15)$$

36. It remains for us to deal with $\mathcal{H}_a$, $5\le a\le 7$.

We have obviously

$$\mathcal{D}_3^1\left(\tfrac{1}{2},\tfrac{1}{2};h\right)\ll\tau(f)\int_{-\infty}^{\infty}\frac{\left|\zeta\left(\frac{1}{2}+ir\right)\right|^4}{|\zeta(1+2ir)|^2}h(r)dr$$
$$\ll\tau(f)K^3(\log K)^5, \quad (36.1)$$

which can of course be replaced by a better bound, but for our purpose this is sufficient. We see that the contribution of $\mathcal{H}_5$ to (35.8) is $\ll K^{\frac{7}{2}}(\log K)^7$, which is negligible in view of (35.2), (35.9), and (35.15).

As to $\mathcal{H}_6$ and $\mathcal{H}_7$, we shall treat the latter only, for the former is analogous and certainly easier than the latter. As we have remarked already, $\mathcal{D}_3^3(u,v;h)$ is regular in the vicinity of $\left(\frac{1}{2},\frac{1}{2}\right)$. Thus we have

$$\mathcal{D}_3^3\left(\tfrac{1}{2},\tfrac{1}{2};h\right) = -\frac{1}{(2\pi)^2}\int_{C_2}\int_{C_1}\frac{\mathcal{D}_3^3(u,v;h)}{\left(u-\frac{1}{2}\right)\left(v-\frac{1}{2}\right)}dudv, \quad (36.2)$$

where

$$C_1: \left|u-\tfrac{1}{2}\right| = \frac{1}{B(2+\log cd)}, \qquad C_2: \left|v-\tfrac{1}{2}\right| = \frac{1}{2B(2+\log cd)}, \quad (36.3)$$

with a sufficiently large constant B. This integrand is, by the explicit formula for $\mathcal{D}_3^3(u,v;h)$ in (33.4),

$$\ll \exp\left(-\tfrac{1}{2}(K/G)^2\right), \quad (36.4)$$

and $\mathcal{H}_7$ is negligible.

Hence we have obtained

Lemma 17. *Let h be as in* (35.5) *with* (35.1)–(35.2). *Then we have, for any fixed c, d with* $\mu(cd)\neq 0$,

$$\sum_{\substack{\kappa_j \\ \kappa_j^2+\frac{1}{4}\in \mathrm{Sp}(\Gamma_0(cd))}} \epsilon_j R_j\left(\tfrac{1}{2},0\right)L_j\left(\tfrac{1}{2};1/c\right)\frac{h(\kappa_j)}{\cosh \pi\kappa_j} \sim \frac{8}{3\pi^{\frac{3}{2}}}GK^3(\log K)^3\prod_{p|c}\left(1-\frac{1}{p^2}\right). \quad (36.5)$$

In particular, if A is fixed, we have

$$\sum_{\kappa}\mathcal{R}(\kappa;A)\frac{h(\kappa)}{\cosh\pi\kappa} \sim \frac{8}{3\pi^{\frac{3}{2}}}GK^3(\log K)^3\sum_{c,d}\mathcal{A}(c,d)\prod_{p|c}\left(1-\frac{1}{p^2}\right), \quad (36.6)$$

where $\frac{1}{4}+\kappa^2\in\bigcup_{c,d}\mathrm{Sp}(\Gamma_0(cd))$ with $\mu(cd)\neq 0$.

Therefore we have established

Theorem. *Provided* $\alpha_n>0$ *for square-free* n *and* $=0$ *otherwise, the function* $Z_2(s;A)$ *has infinitely many simple poles on the line* $\mathrm{Re}\, s=\frac{1}{2}$.

This restriction on the support of α_n will be lifted in our forthcoming work.

Our result suggests that the Mellin transform

$$Z_3(s;1) = \int_1^\infty \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^6 t^{-s}dt \quad (36.7)$$

should have the line $\mathrm{Re}\, s=\frac{1}{2}$ as a natural boundary, for $|\zeta|^6=|\zeta|^4|\zeta|^2$ and $|\zeta|^2$ may be replaced by a finite expression similar to $|A|^2$ via the approximate functional equation. The same was speculated also by a few people other than us, but it appears that our theorem is so far the sole explicit evidence supporting this conjectural assertion. At any event, in view of of REMARK 5 above, it appears reasonable for us to maintain that $Z_3(s;1)$ does not continue beyond the imaginary axis.

This entails

Problems:

(1) Is the set $\bigcup_{q\geq 1}\mathrm{Sp}(\Gamma_0(q))$ dense in the half line $\left(\frac{1}{4},\infty\right)$?

(2) Is the set of κ satisfying (30.3) dense in the positive real axis?

(3) Is the set of κ satisfying (30.3) dense in any half line?

(4) Is the set of κ satisfying (30.3) dense in any interval whose left end point is the origin?

Obviously (1) is to be solved first and (2) must be far more difficult than (1). The third, weaker than (2), appears highly plausible in the light of Lemma 17; on the other hand our method does not seem to extend without new twists so as to include the situation of (4), i.e., the detection of low lying poles.

ADDENDUM. Recently C.P. Hughes and M.P. Young (arXiv:0709.2345 [math.NT]) obtained an asymptotic formula for the mean value (30.1) where the length of A is less than T^η with any fixed $\eta<1/11$. They did not employ the spectral theory of Kloosterman sums. Our method should give a better result than theirs, if it is combined with works by N. Watt on this mean value.

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Ⅰ　趣　旨

この要項は，日本大学理工学部理工学研究所研究ジャーナル刊行内規（以下内規という）第14条に基づき，日本大学理工学部理工学研究所研究ジャーナル（以下研究ジャーナルという）の内容，投稿，執筆等についての必要事項を定める．

Ⅱ　論文内容・投稿資格等について

1　研究ジャーナルの定義

内規第7条に定める研究ジャーナルの内容区分の定義は，次のとおりとする．

① 一般論文とは，通常の意味の一つの独立した原著論文である．

② ノートとは，断片的ではあっても，新しい価値ある事実を含む論文で，著者又は著者以外の既往の論文に対する補遺・意見等も含まれる．

③ 速報とは，独創的で重要な発見又は結論を含み，それを承認するに足りるデータを備え，他に優先して掲載する必要のある論文である．この詳報は，後日，一般論文として投稿することができる．

④ 総合論文とは，著者が発表した複数の原著論文を関連づけ，一連の研究成果としてまとめて執筆したものである．

2　研究ジャーナル特集号の定義（以下特集号という）

特集号は，大学の命による調査団の報告書，その他理工学研究所が必要と認めたものであり，編集は当該調査団等の責任において行う．

3　投稿資格

研究ジャーナルの投稿資格は，次の各号のいずれかに該当する者とする．

① 日本大学理工学部・短期大学部（船橋校舎）（以下学部等という）に在職する者

② 日本大学大学院理工学研究科博士後期課程，前期課程及び日本大学理工学部の在学生（ただし，指導教員の承諾を得なければならない）

③ 多年にわたり専任教員として理工学部に勤務し，現在非常勤講師である者

④ 編集委員会（以下委員会という）が特に認めた者

4　原稿ページ数

研究ジャーナルの内容区分の文字数及び頁数は，以下のとおりとする．

区　　分	一般論文	ノート・速報
文字数	15,000 文字程度	5,000 文字程度
頁数	10 頁程度	4 頁程度

＊　総合論文は，著者と委員会で相談の上決定する．

5　投稿の受付

投稿を希望する者は，所定の理工学研究所研究ジャーナル投稿申請書，掲載論文著作権委譲確認書とともに原稿を研究事務課（以下所管課という）に提出する．

6　原稿の受付及び発行時期

論文誌の発行は年3回とし，原稿の受付及び発行時期は次のとおりとする．ただし，内規第7条及び本要項Ⅱ-1，2に該当しない原稿は，執筆者に返却することがある．

原稿の受付	発行時期
2月末	6月末
6月末	10月末
10月末	翌年の2月末

7　受付年月日

受付年月日は所管課で受付を行った日とし，受理年月日は査読結果に基づき委員会が掲載を決定した日とする．

8　原稿の提出部数等

① 原稿の提出部数は，一般論文，ノート及び総合論文の場合は，正原稿(図，表，写真を含む) 1部並びに複写2部(図，表，写真を含む）とする．

② 速報は，正原稿(図，表，写真を含む) 1部並びに複写(図，表，写真を含む） 1部とする．

③ 前2項はいずれも正原稿の電子データを提出する．

9　論文掲載の採否

論文掲載の採否は，研究ジャーナル刊行内規第11条に基づき委員会が決定する．

10　投稿の取消し

査読過程の修正・内容照会等において，執筆者による修正原稿の提出が依頼の日から2か月以上経過した場合は，最初の原稿受付日を取り消し，再提出された日を新たに原稿受付日とする．ただし，1か年以内に原稿の再提出がない場合は，委員会の議を経て投稿を取り消す．

11　原稿料

原稿料は，支払わない．

12　抜き刷りの給付

執筆者には，研究論文集の抜き刷りを，30部までは無償で給付する．ただし，30部を超える部数を希望する場合は，執筆者が相当分の実費を負担する．

以　上

(内規抜粋)

第11条　論文の掲載の可否は, 査読報告書に基づき審議の上, 委員会が決定する．

2　2名の査読者の査読判定が共に掲載可又は否の場合は，特に問題がなければ判定どおり決定する．

3　2名の査読者間で査読判定が相違した場合，委員会は第3の査読者を選び，査読を依頼し，その結果に基づき委員会で審議をする．

4　査読判定において掲載否の理由が「照会に対する回答不十分」等の場合は，委員会において回答不足項目を検討・審議し，適切な措置をとる．

5　査読判定で研究ジャーナルの種類の変更を求められた場合は, 委員会で検討の上, 著者とその対応を協議・決定する．

6　特集号における論文掲載の可否は，当該調査団等が行うものとする．

以　上

日本大学理工学部理工学研究所研究ジャーナル執筆要項

平成 21 年 4 月 1 日制定
平成 22 年 4 月 1 日改正
平成 22 年 4 月 1 日施行

1　受付原稿

受付原稿は，執筆要項に従って執筆したもので，原則としてMicrosoft Word で作成した電子原稿（以下原稿とする）とする．

2　原稿の標準的作成方法は，次のとおりとする．

① 用紙サイズは，A4 判縦長とし，題名，著者名，概要及びキーワードは，横書き 1 段とし，Microsoft　Word の 43 文字× 38 行を基準とする．本文，参考文献及び付録は，横書き 2 段組とし，Microsoft Word の 20 文字× 38 行 2 段組を基準とする．余白は，上 25 ㎜，下 25 ㎜，左 25 ㎜，右 25 ㎜とする．

② フォントは，邦文においては明朝，欧文は Times New Roman を基本とする．

③ 文字ポイントは，表題を 12pt. とし，それ以外は 10.5pt. とする．邦文はひらがな，カタカナ，漢字を全角とし，欧文英数字は半角を使用する．

④ 邦文論文の順序

(1) 邦文題名
(2) 邦文著者名
(3) 欧文題名
(4) 欧文著者名
(5) 欧文概要
(6) 欧文キーワード
(7) 本文
(8) 参考文献
(9) 付録

⑤ 欧文論文の順序

(1) 欧文題名
(2) 欧文著者名
(3) 欧文概要
(4) 欧文キーワード
(5) 本文
(6) 参考文献
(7) 付録

3　第 1 ページの体裁

① 邦文題名は，中央揃えで記載する．

② 邦文著者名は，題名から 1 行開けて中央揃えで記載する．名前の後には，著者の所属を参照するために，上付きで記号を付記し，脚注（後述）で所属を記載する．

③ 欧文題名は，邦文著者名から 1 行あけて中央揃えで記載する．

④ 欧文著者名は，欧文題名から 1 行あけて中央揃えで記載する．なお，著者が複数の場合，最後の著者名とその直前名の間は and で区切り，それ以外はコンマで区切る．

⑤ 概要は，欧文著者名から 1 行あけて，強調文字の英文で中央に Abstract と書き，200words 程度からなる概要本文を記載する．

⑥ キーワードは，概要から 1 行あけて，英文で Key Words: の文字列に続き，5words 以内で記載する．

⑦ 本文は，キーワードから 1 行あけて記載する．

⑧ 著者の所属は，脚注に次のとおり記載する．

邦文の場合

* 日本大学理工学部物質応用化学科；日本大学理工学部理工学研究所材料創造研究センター

欧文の場合，英文なら

* Department of Materials and Applied Chemistry, College of Science and Technology, Nihon University ; The Center for Creative Materials Research, Research Institute of Science and Technology, College of Science and Technology, Nihon University

4　本文の体裁

① 章・節・項は，次のとおりとする．本文は，それぞれから 1 行改行して記載する．

	表記	表示位置
章	1.　2.　3.	行の中央
節	1.1　1.2　1.3	行の左端から 1 文字目
項	1.1.1　1.1.2　1.1.3	

② 句読点

邦文は，全角カンマ（，）と全角ピリオド（．）を使用する．
欧文は，半角カンマ(,)と半角ピリオド(.)を使用する．

③ 数字

(1) 原則として算用数字（アラビア文字）（半角）を使用する．
(2) 熟語，成句，固有名詞は漢数字を使用する．
(3) 第一に，第二に，一つ目，二つ目などは，論文中で漢数字又は算用数字（半角）で統一する．

④ 図と表

(1) 図及び表は，縮尺を考慮した完全な図面として文中に挿入する .
(2) 図（グラフ，説明図，写真等）は，図 1，図 2 として，その次に図の表題を記載する．図の番号及び表題は，図の下に記載することを原則とする．
(3) 表は，表 1，表 2 としてその次に表の表題を記載する．表の番号及び表題は，表の上に記載することを原則とする．
(4) グラフの座標軸の説明は横書きで，縦軸は下から上へ，横軸は左から右へそれぞれ中央に記載する．

例：撹拌トルク□ T □〔N・m〕
Distance from wall □ y □〔cm〕※□印はひとコマあける意味

(5) 図表等を他の文献から転載する場合は，著者の責任において転載許可を得て，その出展を明記すること．

⑤ 用語はそれぞれ学会で決められたもの，又は日本工業規格（JIS）の標準用語を用いる．また付録Ⅱ用字例も参考とすること．

⑥ 単位は，国際単位すなわち SI（Systeme International d’Unites）による．単位記号については，それぞれの学会で制定したもの，又は JISZ8202（1974），前述の SI，若しくは DIN1304 Allgemeine Formelzeichen（1968）を参照する．

⑦ 参考文献の表記

(1) 文献の本文の引用箇所に，右肩 1)・2)・5)・~8) のように片カッコを付して番号を記載する．
(2) 表記は番号順に列記すること．

⑧　記述上の注意

（1）文章は文章的口語体とし，特に欧文又はカタカナ書きを必要とする部分以外は，漢字・かな(ひらがな)まじり書きとする.

（2）漢字は常用漢字のみを使用するものとするが，常用漢字であっても表外音訓は使用しない．ただし，文脈上どうしても常用漢字以外の漢字を使用しなければならない場合は，ルビをふるものとする.

（3）かなは，新かなづかいによる．ただし，外来語はカタカナ書きとする.

5　参考文献の体裁

①　文献の番号は､1 論文ごとに通し番号とし，片カッコを付して番号を記載する.

②　同一の著者が同年に複数の著書又は論文を発表している場合，文献は，発行の古い順から表記する .

③　邦文文献の表記

・(論文の場合)：著者名（発行年）：“論文名”，書物名又は雑誌名，巻数，号数，ページ数.

（例）

1）加鳥　裕明 （2002）：“圧電積層平板の有限要素解析”，日本機械学会論文集 A編，第 68 巻，第 666 号，pp.189-195.

・(単行本の場合)：著者名（発行年）：書物名，発行所.

（例）

2）山内 鴻廣隆（2003）：環境の倫理学，丸善株式会社.

④　欧文文献の表記

（1）著者名は，単著の場合は，苗字，名前のイニシャル.

（2）著者名が複数名の場合は，1 番目の著者の苗字，名前のイニシャル.，２番目の著者のイニシャル . 苗字 and 最後の著者のイニシャル. 苗字(苗字の後には，ピリオドなし)

・(論文の場合)：著者名（発行年）：“論文名”，書物名又は雑誌名（イタリック），巻数，号数，ページ数.

（例）

1）Craig, J. (1999) : “Weight Estimates and Control”, in G. A. Khoury and J. D. Gillett (eds.), Airship Technology, Cambridge, Cambridge University Press, pp.235-271.

2）Potvin, J., G. peek and B. Brocato (2003) : “New Model of Decelerating Bluff-Body Drag”, *Journal of Aircraft*, Vol.40, No.2, pp370-377.

・(単行本の場合)：著者名（発行年）：書物名，発行地，発行所.

（例）

3）McRuer, D., I. Ashkenas and D. Graham (1973) : Aircraft Dynamics and Automatic Control, Princeton, Princeton University Press.

・(WWW の場合)：ブラウザー名（発行年）：書物名，www アドレス.

（例）

4）Selig, M. S. (1998) : “UIUC Airfoil Coordinates Database”, UIUC Airfoil Date Site, URL : http://www.ae.illinois.edu/m-selig

⑤　本研究ジャーナルの欧文表記の略称

J. Res. Inst. Sci. Tech., Nihon Univ.

以　上

平成　　年　　月　　日

第　　号　理工学研究所研究ジャーナル投稿申請書

理工学研究所長　殿

申請者　所属＿＿＿＿＿＿＿＿＿＿＿＿

氏名＿＿＿＿＿＿＿＿＿＿＿＿印

1　表　　題：和文＿＿＿＿＿＿＿＿＿＿＿＿＿＿＿＿＿＿＿＿＿＿

英文……………………………………………………………

2　原稿の種類：　一般論文　　ノート　　速報　　総合論文　（該当するものに○）

3　原稿の枚数：　本文＿＿＿枚　　図＿＿＿枚（　　）　写真＿＿＿枚（　　）　表＿＿＿枚（　　）

カラー印刷とする図表類のある場合は外数で（　）に枚数を記入して下さい。

超過ページ，カラー印刷のある場合は実費を請求する場合があります。

4　著者情報：※連名者欄が不足の場合は，適宜追加してご記入ください。

コレスポンディングオーサーには，「*」記号を付けて下さい。

「　」第１著者名＿＿＿＿＿＿＿＿＿＿＿＿ローマ字（　　　　　　　　　　　　）

所属（　　　　　　　　　　　　）　　資格（　　　　　　　　　　　　）

Organization（　　　　　　　　　）　　Title（　　　　　　　　　　　　）

E-mail（　　　　　　　　　　　　）　　Tel.（　　　　　　　　　　　　）

「　」第２著者名＿＿＿＿＿＿＿＿＿＿＿＿ローマ字（　　　　　　　　　　　　）

所属（　　　　　　　　　　　　）　　資格（　　　　　　　　　　　　）

Organization（　　　　　　　　　）　　Title（　　　　　　　　　　　　）

E-mail（　　　　　　　　　　　　）　　Tel.（　　　　　　　　　　　　）

「　」第３著者名＿＿＿＿＿＿＿＿＿＿＿＿ローマ字（　　　　　　　　　　　　）

所属（　　　　　　　　　　　　）　　資格（　　　　　　　　　　　　）

Organization（　　　　　　　　　）　　Title（　　　　　　　　　　　　）

E-mail（　　　　　　　　　　　　）　　Tel.（　　　　　　　　　　　　）

5　希望査読者

※日本大学以外の査読者を５名程度推薦願います。査読者選定の参考にさせていただきます。

	氏　名	所　属	資　格	連　絡　先
1				住　所　〒 Tel. E-mail
2				住　所　〒 Tel. E-mail
3				住　所　〒 Tel. E-mail
4				住　所　〒 Tel. E-mail
5				住　所　〒 Tel. E-mail

6　抜刷り：30 部までは抜刷りを無料で給付します。追加部数をご希望の場合（実費負担）：______部

7　申請者の指導教員として，上記の執筆者の投稿を承認いたします。＜※投稿者が学生の場合のみ＞

所属：________________________________　　　　氏名__________________________印

研究事務課受付日：平成　　年　　月　　日

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